LUZIN GAPS

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ABSTRACT. We isolate a class of $F_{\sigma\delta}$ ideals on $\mathbb{N}$ that includes all analytic P-ideals and all F_σ ideals, and introduce 'Luzin gaps' in their quotients. A dichotomy for Luzin gaps allows us to freeze gaps, and prove some gap preservation results. Most importantly, under PFA all isomorphisms between quotient algebras over these ideals have continuous liftings. This gives a partial confirmation to the author's rigidity conjecture for quotients $\mathcal{P}(\mathbb{N})/\mathcal{I}$. We also prove that the ideals NWD($\mathbb{Q}$) and NULL($\mathbb{Q}$) have the Radon–Nikodým property, and (using OCA_∞) a uniformization result for $\mathcal{K}$-coherent families of continuous partial functions.

One of the most fascinating facts about the Boolean algebra $\mathcal{P}(\mathbb{N})/\mathrm{Fin}$ was discovered by Hausdorff in 1908. In [20], he constructed two families $\mathcal{A}$ and $\mathcal{B}$ of sets of integers such that

(a) $A \cap B$ is finite for all $A \in \mathcal{A}$ and all $B \in \mathcal{B}$,
(b) for every $C \subseteq \mathbb{N}$ either $A \setminus C$ is infinite for some $A \in \mathcal{A}$ or $B \cap C$ is infinite for some $B \in \mathcal{B}$, and
(c) both and $\mathcal{A}$ and $\mathcal{B}$ have order-type equal to ω_1 with respect to the inclusion modulo finite.

Families $\mathcal{A}$ and $\mathcal{B}$ satisfying (a) are *orthogonal*, those satisfying both (a) and (b) form a *gap* in $\mathcal{P}(\mathbb{N})/\mathrm{Fin}$ (or, they are *not separated* over Fin, the ideal of finite sets), and if they furthermore satisfy (c) they form an (ω_1, ω_1)-*gap*. If (a) and (b) hold and both $\mathcal{A}$ and $\mathcal{B}$ are linearly ordered by the inclusion modulo finite, a gap is *linear*. A gap (linear or not) is *Hausdorff* if both of its sides are σ-directed under the inclusion modulo finite.

Another major advance in the study of gaps in $\mathcal{P}(\mathbb{N})/\mathrm{Fin}$ was made by Kunen ([29]), who used a condition originally isolated by Luzin in [32] to prove that for every (ω_1, ω_1)-gap there is a ccc poset that freezes it. (A gap is *indestructible*, or *frozen*, if it remains a gap in every $\aleph_1$-preserving forcing extension.) Remarkably, a linear gap can be frozen by an $\aleph_1$-preserving forcing if and only if it contains an (ω_1, ω_1)-gap. Kunen used freezing to prove that $\mathcal{P}(\mathbb{N})/\mathrm{Fin}$ need not be universal for linear orderings of size at most $2^{\aleph_0}$ even if Martin's Axiom, MA, is assumed. The technique of freezing gaps has played an important role in a variety of subjects, from automatic continuity in Banach algebras ([6]) to the study of automorphisms of $\mathcal{P}(\mathbb{N})/\mathrm{Fin}$ ([35]).

Todorčević ([38, §8]) has associated to every gap $(\mathcal{A}, \mathcal{B})$ on $\mathbb{N}$ (linear or not) an open partition of the restricted product $\mathcal{A} \otimes B = \{(A, B) \in \mathcal{A} \times \mathcal{B} : A \cap B = \emptyset\}$. This partition in particular detects whether the gap is destructible or not. Largely for this purpose Todorčević has formulated a dichotomy about open colorings known today as the Open Coloring Axiom, or OCA. It should be pointed out that OCA is equivalent to a statement about freezing nontrivial coherent families of functions, very closely related to gaps [11, Proposition 2.2.11]. OCA gave impetus to later progress on understanding the structure of analytic quotients.

In [34] Shelah has found a forcing extension in which all automorphisms of $\mathcal{P}(\mathbb{N})/\operatorname{Fin}$ are trivial (or in our terminology, they have a completely additive lifting; see §12). The key lemma in the proof is that a nontrivial automorphism Φ can be destroyed by generically adding $X \subseteq \mathbb{N}$ such that the families (by Φ_* we denote the lifting of Φ, see §5) $\{\Phi_*(A) : A \subseteq^* X, A \in V\}$ and $\{\Phi_*(B) : B \cap X \in \operatorname{Fin}, B \in V\}$ form a gap. In order to preserve such gaps while destroying other nontrivial automorphisms, Shelah has developed a technique of *oracle-chain condition*. In [35] it was proved that the Proper Forcing Axiom, PFA, implies that all automorphisms of $\mathcal{P}(\mathbb{N})/\operatorname{Fin}$ are trivial. One of the key parts of this proof consisted in making a given gap in $\mathcal{P}(\mathbb{N})/\operatorname{Fin}$ indestructible. New evidence that the ability to freeze the gaps is crucial in this proof was provided by Veličković ([44]), who proved that OCA and MA already imply that all automorphisms of $\mathcal{P}(\mathbb{N})/\operatorname{Fin}$ are trivial.

In [11, §3.3, 'OCA lifting theorem'], the author extended Veličković's result (more precisely, the part of it that corresponds to Conjecture 2 below) to all quotients over analytic P-ideals. It was then natural to expect that Kunen's result on the structure of gaps would soon be extended to all quotients over analytic P-ideals. It therefore came as a surprise when the author constructed an analytic Hausdorff gap $\mathcal{A}, \mathcal{B}$ in a quotient over every dense F_σ P-ideal $\mathcal{I}$ ([14], see also [11, Theorem 5.10.2]). Such a gap cannot be made indestructible, since a result of Todorčević ([40]) easily implies that increasing the additivity of Lebesgue measure separates such gaps (see also [11, Corollary 5.10.3]).

The motivation for this paper comes from the rigidity conjectures below. PFA stands for the Proper Forcing Axiom, see [38]. See §5 for our definition of lifting.

Conjecture 1. *PFA implies that every isomorphism between analytic quotients has a completely additive lifting.*

This would imply that every analytic quotient is uniquely determined by the underlying analytic ideal. Some instances of Conjecture 1 were proved from weaker axioms, OCA and MA, in [11] (see also [10]). Conjecture 1 and some of its variants are discussed in [15], where it was shown to be equivalent to the conjunction of the following two conjectures (see §8 for definitions).

Conjecture 2. *PFA implies that every isomorphism between analytic quotients has a continuous lifting.*

Conjecture 3. *Every isomorphism between analytic quotients that has a continuous lifting has a completely additive lifting.*

Conjecture 2 was verified for all analytic P-ideals in [11]. Conjecture 3 was verified for all *nonpathological* analytic P-ideals (see Definition 2.1) in [11] and for many other Borel ideals ([26, 27]). Similar rigidity phenomena have been observed in the context of other quotient structures, such as groups and lattices (see [13]). Forcing

and gaps are relevant only to Conjecture 2, since the statement of Conjecture 3 is absolute between transitive models of set theory that contain all countable ordinals.

We will isolate a class of ideals 'strongly countably determined by closed approximations' (see Definition 2.2) and develop a technique of freezing gaps in their quotients (Theorem 4.4). We will prove that every gap in a quotient over an ideal in this class either is separated by closed sets (in a rather weak sense that will be made more precise in Definition 3.4) or it can be frozen. The condition we use to assure the indestructibility generalizes the one used by Kunen, Luzin and Todorčević. The class of ideals strongly countably determined by closed approximations includes all analytic P-ideals, all F_σ ideals, as well as all other $F_{\sigma\delta}$ ideals known to the author (see Definition 2.5).

Our freezing technique is applied within an extension of the Shelah–Steprāns proof to verify Conjecture 2 for this class of ideals. We will moreover prove that all strongly countably determined ideals have the continuous lifting property, introduced in [15] (Definition 9.3). In particular, we prove the following.

Theorem 4 (PFA)**.** *If $\mathcal{I}$ and $\mathcal{J}$ are analytic ideals and at least one of them is strongly countably determined by closed approximations, then every isomorphism between their quotients has a continuous lifting.*

Proof. This is a special case of Theorem 10.4, see Corollary 10.5. $\square$

This gives another proof (from stronger assumptions) of the 'OCA lifting theorem' for analytic P-ideals of [11]. As an illustration of the new technique of freezing gaps, we extend some results known for $\mathcal{P}(\mathbb{N})/\mathrm{Fin}$ to quotients over any ideal $\mathcal{I}$ that is strongly countably determined by closed approximations (see [38, §8]).

Theorem 5 (PFA)**.** *If $\mathcal{I}$ is strongly countably determined by closed approximations, then $\mathcal{P}(\mathbb{N})/\mathcal{I}$ is not universal for linear orderings of size $2^{\aleph_0}$. Also, every complete Boolean algebra that embeds into $\mathcal{P}(\mathbb{N})/\mathcal{I}$ has to be ccc.*

Proof. This is Theorem 11.1, because every complete Boolean algebra that is not ccc contains a copy of $\mathcal{P}(\omega_1)$. $\square$

Regarding Conjecture 3, in §12 we prove the following (see Definition 2.5).

Theorem 6. *Every homomorphism of $\mathcal{P}(\mathbb{N})$ into a quotient over $\mathrm{NWD}(\mathbb{Q})$ or $\mathrm{NULL}(\mathbb{Q})$ that has a continuous lifting has a completely additive lifting.*

Proof. This is Theorem 12.1. $\square$

Nonpathological ideals are defined in Definition 2.1.

Corollary 7. *Conjecture 1 is true for the class of ideals containing all* (i) *nonpathological ideals, as well as the ideals* (ii) $\mathrm{NWD}(\mathbb{Q})$, (iii) $\mathrm{NULL}(\mathbb{Q})$ *and* (iv) $\mathcal{Z}_W$.

Proof. By Lemma 2.4 and Lemma 2.6, all of these ideals are strongly countably generated by closed approximations. By Theorem 4, under PFA every isomorphism of an analytic quotient with a quotient over such an ideal has a continuous lifting. Let Φ be an isomorphism between a quotient over any of these ideals with a continuous lifting. By [11] (i) (for P-ideals), [26] ((i) for F_σ ideals, (iv)), and Theorem 6 ((ii), (iii)), both Φ and its inverse have a completely additive lifting. By Lemma 12.5, such an isomorphism is induced by a Rudin–Keisler isomorphism between the underlying ideals. $\square$

The same proof gives the following (cf. [11, Corollary 3.4.6 and §3.6]).

Corollary 8 (PFA). *All automorphisms of a quotient over any nonpathological ideal,* NWD($\mathbb{Q}$)*,* NULL($\mathbb{Q}$) *and* $\mathcal{Z}_W$ *are trivial.*   □

The last two sections of this paper depend only on §§2-4, and they illustrate uses of Luzin gaps from a different point of view. In [39] Todorčević proved that there are no analytic Hausdorff gaps in $\mathcal{P}(\mathbb{N})/\mathrm{Fin}$. This strengthens the classical Hurewicz separation principle. In [41], Todorčević used his separation principle to prove that all Hausdorff gaps are preserved by those embeddings of $\mathcal{P}(\mathbb{N})/\mathrm{Fin}$ into analytic quotients that have a Baire-measurable lifting. In [11, §5.9] this was extended to arbitrary embeddings of $\mathcal{P}(\mathbb{N})/\mathrm{Fin}$ into analytic quotients, assuming OCA and MA. By [14], there are analytic Hausdorff gaps in quotients over some F_σ P-ideals, and such gaps are not preseved even by embeddings into quotients over some other F_σ P-ideals. However, in §13 we will give a separation principle for analytic gaps in quotients over an arbitrary ideal strongly countably determined by closed approximations. This principle is strong enough to imply Todorčević's result in the case of Fin, and to give the following.

Theorem 9. *There are no analytic Hausdorff Luzin gaps in a quotient over any ideal strongly countably determined by closed approximations.*

Proof. This is a consequence of Corollary 13.6.   □

In §14 we will see to what extent the gaps are preserved by embeddings of quotients over ideals that are strongly countably determined by closed approximations.

Organization of the paper. In §1 we prove a new uniformization result for coherent families of continuous functions.

In §2 we introduce countably determined ideals and strongly countably determined ideals, and show that they include all F_σ ideals, all analytic P-ideals and some other $F_{\sigma\delta}$ ideals. The key objects of study in this paper, Luzin gaps, aloof gaps and unapproachable gaps, are defined in §3. In §4 we prove a dichotomy result for gaps that is a basis for our freezing technique. Instead of a lifting of a homomorphism $\Phi\colon \mathcal{P}(\mathbb{N}) \to \mathcal{P}(\mathbb{N})/\mathcal{I}$, in §5 we consider functions that approximate Φ with respect to a given closed approximation $\mathcal{K}$ to $\mathcal{I}$, and in particular those whose graphs are covered by graphs of countably many Baire-measurable functions. In §6 we redefine the notion of an almost lifting from [11], and in §7 we continue the work started in §5. One of the key technical difficulties in extending the OCA lifting theorem to a class wider than the analytic P-ideals is dealt with in §8. Here we find a representation of a homomorphism $\Phi\colon \mathcal{P}(\mathbb{N}) \to \mathcal{P}(\mathbb{N})/\mathcal{I}$ that respects a given closed approximation $\mathcal{K}$ to $\mathcal{I}$ (Lemma 8.4). Such representations cohere with respect to $\mathcal{K}$ (Lemma 8.6).

In §9 we prove a local version of Theorem 4, and in §10 we complete the proof of Theorem 4. We actually prove a version of Theorem 4 for arbitrary homomorphisms, the *continuous lifting property* for strongly countably determined ideals.

In §11 we prove Theorem 5 using only results of §§3 and 4. In §12 we prove Theorem 6, using only results of §8. In §§13 and 14 we consider separation and preservation of gaps, respectively. We conclude with a short list of open problems in §15.

1. Uniformization of $\mathcal{K}$-coherent families

In the present section we will prove a new uniformization result for a family of functions that are coherent in some way (cf. [38, Theorem 8.7], [44, Theorem 4.2 and remark on p. 9], [42, Theorem 10.3 and its variations, Theorem 10.6], [11, Chapter 2]). A coherent family of partial functions is usually a family of functions $f_A \colon A \to \mathbb{N}$, where A ranges over some index set (typically an ideal on $\mathbb{N}$). It is required that for all A and B the functions f_A and f_B differ on at most finitely many places on $A \cap B$. Such a family is trivial if there is a single function $g \colon \mathbb{N} \to \mathbb{N}$ such that f_A and $g \restriction A$ differ on at most finitely many elements for every A. An overview of this notion and known uniformization results, tightly connected to Todorčević's Open Coloring Axiom (see §4), is given in [11, Chapter 2].

For an ideal $\mathcal{I}$ we may define a family of $\mathcal{I}$-coherent partial functions $f_A \colon A \to \mathbb{N}$ by requiring that $\{n \in A \cap B | f_A(n) \neq f_B(n)\} \in \mathcal{I}$ for all A and B. Hence a coherent family is the same as a Fin-coherent family. The well-developed theory of Fin-coherent families does not carry over to this more general context (see [11, Theorem 2.5.1]). In this section we will be considering an even more general context, families of continuous functions $f_A \colon \mathcal{P}(A) \to \mathcal{P}(\mathbb{N})$ that are $\mathcal{I}$-coherent in some way, to be made precise later.

1.1. **Open colorings.** Recall that $[X]^2 = \{\{x, y\} | (x, y) \in X^2 \text{ and } x \neq y\}$. If $K \subseteq [X]^2$, then a set $B \subseteq X$ is K-*homogeneous* if $[B]^2 \subseteq K$. It is σ-K-*homogeneous* if it can be covered by countably many K-homogeneous sets.

If X is a topological space, then $K \subseteq [X]^2$ is *open* if $\{(x, y) | \{x, y\} \in K_0\}$ is an open subset of X^2. A partition $[X]^2 = K_0 \cup K_1$ is said to be *open* if there is a separable metric topology on X such that K_0 is an open subet of $[X]^2$. This is equivalent to saying that $K_0 = \bigcup_{i=1}^{\infty} U_i \times V_i$ for some U_i, V_i included in X (see, e.g., [11, Chapter 2]). The topology that makes K_0 open is frequently finer than the natural topology on X (see [11, p. 63]; also compare Definition 1.3 and the paragraph following it).

OCA. If X is a separable metric space and $[X]^2 = K_0 \cup K_1$ is an open partition, then one of the following applies:

    (a) X is σ-K_1-homogeneous, or
    (b) X has an uncountable K_0-homogeneous subset.

By a result of Todorčević, OCA is a consequence of PFA (see, e.g., [42]). Two strengthenings of OCA were introduced in [9]. The name OCA_∞ was used for the weaker one, which will not be used in the present paper.

OCA_∞. If X is a separable metric space and $[X]^2 = K_0^n \cup K_1^n$ ($n \in \mathbb{N}$) is a sequence of open partitions such that $K_0^{n+1} \subseteq K_0^n$ for all n, then one of the following applies:

    (1) There are F_n ($n \in \mathbb{N}$) such that $X = \bigcup_{n=1}^{\infty} F_n$ and $[F_n]^2 \subseteq K_1^n$ for every n.
    (2) There are an uncountable $\mathcal{Z} \subseteq 2^{\mathbb{N}}$ and a continuous injection $f \colon \mathcal{Z} \to X$ such that for all distinct x, y in $\mathcal{Z}$ we have

$$\{f(x), f(y)\} \in K_0^{\Delta(x,y)}$$

    (where $\Delta(x, y) = \min\{n | x(n) \neq y(n)\}$).

Thus OCA is a special case of OCA_∞, when $K_0^n = K_0$ for all n. In [9, §3] it was proved that PFA implies OCA_∞, as (a) from [9, §3] implies (1) and (b') of [9, §3] implies (2). It is not known whether OCA and OCA_∞ are equivalent statements.

Definition 1.1. Given X and a decreasing sequence $L(n)$ ($n \in \mathbb{N}$) of subsets of $[X]^2$ and nonincreasing $h \colon \mathbb{N} \to \mathbb{N}$ such that $\lim_n h(n) = \infty$, we say that a set $Y \subseteq X$ is *strongly $L(h)$-homogeneous* if there is an injection $g \colon Y \to \mathcal{P}(\mathbb{N})$ such that
$$\{x, y\} \in L(h(\Delta(g(x), g(y))))$$
for all $\{x, y\} \in [Y]^2$. A set is *strongly L-homogeneous* if it is strongly $L(h)$-homogeneous for some nondecreasing h such that $\lim_n h(n) = \infty$.

Thus clause (2) of the statement of OCA_∞ says that X has an uncountable strongly K_0-homogeneous subset. Note that every strongly L-homogeneous set can be covered by finitely many $L(n)$-homogeneous sets for every $n \in \mathbb{N}$.

Following [26], for a partial function $f \colon \mathcal{P}(\mathbb{N}) \to \mathcal{P}(\mathbb{N})$ and $s, t \in \mathrm{dom}(f)$ let
$$D_{st}^f = f(s)\Delta f(t)\Delta f(s\Delta t).$$

Definition 1.2. Let $\mathcal{K}$ be a closed subset of $\mathcal{P}(\mathbb{N})$. If $\mathcal{J} \subseteq \mathcal{P}(\mathbb{N})$, then a family $\mathcal{F} = \{f_A | A \in \mathcal{J}\}$ is a *$\mathcal{K}$-coherent family of continuous functions indexed by $\mathcal{J}$* if for all A and B in $\mathcal{J}$ we have

(1) $f_A \colon \mathcal{P}(A) \to \mathcal{P}(\mathbb{N})$ is continuous,
(2) $D_{st}^f \in \mathcal{K}$ for all $s, t \subseteq A$, and
(3) there is $n = n(A, B)$ such that
$$(f_A(s)\Delta f_B(s)) \setminus n \in \mathcal{K}$$

   for all $s \subseteq (A \cap B) \setminus n$.

Definition 1.3. For $\mathcal{F} = \{f_A \colon \mathcal{P}(A) \to \mathcal{P}(\mathbb{N}) | A \in \mathcal{J}\}$, $\mathcal{K} \subseteq \mathcal{P}(\mathbb{N})$, $D \subseteq \mathbb{N}$ and $n \in \mathbb{N}$, define a partition $[\mathcal{J}]^2 = L_0^{\mathcal{K}}(D, n) \cup L_1^{\mathcal{K}}(D, n)$ by

(4) $\{A, B\} \in L_0^{\mathcal{K}}(D, n)$ if and only if there is $s \subseteq (A \cap B \cap D) \setminus n$ such that
$$(f_A(s)\Delta f_B(s)) \setminus n \notin \mathcal{K}.$$

So condition (3) of Definition 1.2 says that for some $n = n(A, B)$ we have $\{A, B\} \in L_1^{\mathcal{K}}(\mathbb{N}, n)$. Note that if $\mathcal{L}$ is closed and the functions f_A are continuous, then there is a separable metric topology on $\mathcal{F}$ such that $L_0^{\mathcal{L}}(D, n)$ is an open subset of $[\mathcal{F}]^2$.

Lemma 1.4 (OCA_∞). *Assume $\mathcal{L}$ is closed and $\mathcal{F}$ is an $\mathcal{L}$-coherent family of continuous functions indexed by a P-ideal $\mathcal{J}$. Then one of the following applies:*

(5) *There are $\mathcal{X}_n$ ($n \in \mathbb{N}$) such that $\mathcal{J} = \bigcup_n \mathcal{X}_n$ and $[\mathcal{X}_n]^2 \subseteq L_1^{\mathcal{L}}(\mathbb{N}, n)$ for all n.*
(6) *The ideal $\mathcal{J}$ has an uncountable strongly $L_0^{\mathcal{L}}(\mathbb{N})$-homogeneous subset.*

Proof. We have $L_0^{\mathcal{L}}(\mathbb{N}, n) \supseteq L_0^{\mathcal{L}}(\mathbb{N}, n+1)$, and these partitions are open. If (5) fails, then by OCA_∞ there are an uncountable $\mathcal{Z} \subseteq \mathcal{P}(\mathbb{N})$ and an injection $g \colon \mathcal{Z} \to \mathcal{J}$ such that for all $x \neq y$ in $\mathcal{Z}$ we have
$$\{g(x), g(y)\} \in L_0^{\mathcal{L}}(\mathbb{N}, \Delta(x, y)).$$
Then g^{-1} witnesses that $\mathcal{X}$ is strongly $L_0^{\mathcal{L}}(\mathbb{N}, \mathrm{id})$-homogeneous. $\qquad\square$

For $\mathcal{K} \subseteq \mathcal{P}(\mathbb{N})$, write $\mathcal{K}^1 = \mathcal{K}$ and $\mathcal{K}^{n+1} = \{s \cup t | s \in \mathcal{K}^n, t \in \mathcal{K}\}$.

Lemma 1.5 (OCA). *Assume $\mathcal{F}, \mathcal{K},$ and $\mathcal{J}$ are as in Definition 1.3 and $\mathcal{X} \subseteq \mathcal{J}$ is strongly $L_0^{\mathcal{K}^3}$-homogeneous. Then there is an uncountable $\mathcal{Y} \subseteq \mathcal{J}$ that is strongly $L_0^{\mathcal{K}}$-homogeneous and forms an increasing ω_1-chain under $\subseteq^*$.*

Proof. Let $\mathcal{X} = \{A_\alpha | \alpha < \omega_1\}$ and let h be such that $\mathcal{X}$ is strongly $L_0^{\mathcal{K}^3}(h)$-homogeneous. Pick $B_\alpha \in \mathcal{J}$ so that $A_\xi \subseteq^* B_\alpha$ for $\xi < \alpha$ and $A_\alpha \subseteq B_\alpha$. Let $g : \mathcal{X} \to \mathcal{P}(\mathbb{N})$ be as in Definition 1.1. Let $g_1(B_\alpha) = g(A_\alpha)$ and $p(B_\alpha) = A_\alpha$ for all $\alpha < \omega_1$. By OCA, g_1 is continuous on an uncountable set (see [38, Proposition 8.4b]), so we may assume g_1 is continuous.

Find $\bar{n}$ and an uncountable $I \subseteq \omega_1$ such that

$$\{A_\alpha, B_\alpha\} \in L_1^{\mathcal{K}}(\mathbb{N}, \bar{n})$$

for all $\alpha \in I$. By refining I further, we may assume that

$$h(\Delta(g_1(B_\alpha), g_1(B_\beta))) \geq \bar{n}$$

for all distinct α and β in I.

Claim 1.6. *The set $\mathcal{Y} = \{B_\alpha | \alpha \in I\}$ is strongly $L_0^{\mathcal{K}}(h)$-homogeneous.*

Proof. Pick distinct z and y in $\mathcal{Y}$. Let us write

$$\bar{\Delta}(y, z) = h(\Delta(g_1(y), g_1(z))).$$

There is $s \subseteq (p(z) \cap p(y)) \setminus \bar{\Delta}(y, z)$ such that

$$(f_{p(z)}(s) \Delta f_{p(y)}(s)) \setminus \bar{\Delta}(y, z) \notin \mathcal{K}^3.$$

But $s \subseteq p(z) \cap p(y)$, and for $x \in \{y, z\}$ we have

$$(f_x(s) \Delta f_{p(x)}(s)) \setminus \bar{n} \in \mathcal{K}.$$

Since $\bar{n} \leq \bar{\Delta}(y, z)$, we must have

$$f_z(s) \Delta f_y(s)) \setminus \bar{\Delta}(y, z) \notin \mathcal{K},$$

and since y and z were arbitrary elements of $\mathcal{Y}$, this proves the claim. $\square$

Since $\mathcal{Y}$ is an ω_1-chain under $\subseteq^*$, this concludes the proof of Lemma 1.5. $\square$

Lemma 1.7. *Assume $\mathcal{F}, \mathcal{K}$ and $\mathcal{J}$ are as in Definition 1.3 and $\mathcal{X} \subseteq \mathcal{J}$ is strongly $L_0^{\mathcal{K}^2}$-homogeneous and well-ordered in type ω_1 by $\subseteq^*$. Then for any two uncountable subsets $\mathcal{V}$ and $\mathcal{W}$ of $\mathcal{X}$ and any $n \in \mathbb{N}$ there are uncountable $\mathcal{V}' \subseteq \mathcal{V}$ and $\mathcal{W}' \subseteq \mathcal{W}$ such that*

$$\mathcal{V}' \times \mathcal{W}' \subseteq L_0^{\mathcal{K}}(\mathbb{N}, n).$$

Proof. Let h be such that $\mathcal{X}$ is strongly $L_0^{\mathcal{K}^2}(h)$-homogeneous. Let $\mathcal{V} = \{A_\alpha | \alpha < \omega_1\}$ and $\mathcal{W} = \{B_\alpha | \alpha < \omega_1\}$ be $\subseteq^*$-increasing enumerations. We may assume that $A_\alpha \subseteq^* B_\alpha$ for all α. By going to an uncountable subset and re-enumerating our sequences, we may find $\bar{m}$ such that for all α we have

(7) $A_\alpha \setminus B_\alpha \subseteq \bar{m}$,

(8) $\{A_\alpha, B_\alpha\} \in L_1^{\mathcal{K}}(\mathbb{N}, \bar{m})$, and

(9) for every $k \in \mathbb{N}$ the set of all $\alpha < \omega_1$ satisfying

    (a) $A_\alpha \cap k = A_\xi \cap k$,

    (b) $B_\alpha \cap k = B_\xi \cap k$,

    (c) $f_{A_\alpha}(s) \cap k = f_{A_\xi}(s) \cap k$ for all $s \subseteq A_\alpha \cap k$, and

    (d) $f_{B_\alpha}(s) \cap k = f_{B_\xi}(s) \cap k$ for all $s \subseteq B_\alpha \cap k$,

    is uncountable.

Clause (9) says that the sets $\{A_\alpha | \alpha < \omega_1\}$ and $\{B_\alpha | \alpha < \omega_1\}$ are both $\aleph_1$-dense in a suitable separable metric topology that makes the pertinent partition open. Since $\mathcal{X}$ is strongly $L_0^{\mathcal{K}^2}(h)$-homogeneous, there are $\alpha \neq \beta$ such that for some $s \subseteq (A_\alpha \cap A_\beta) \setminus \bar{m}$ we have

$$(f_{A_\alpha}(s) \Delta f_{A_\beta}(s)) \setminus \bar{m} \notin \mathcal{K}^2.$$

By (7), we have $s \subseteq B_\alpha \cap B_\beta$. By (8), we must have $f_{A_\alpha}(s) \Delta f_{B_\beta}(s) \setminus \bar{m} \notin \mathcal{K}$. By the continuity of f_{A_α}, f_{B_β}, and (9), there are uncountable $\mathcal{U}'$ and $\mathcal{W}'$ as required. $\square$

Theorem 1.8 (OCA$_\infty$+MA). *Assume $\mathcal{K} \subseteq \mathcal{P}(\mathbb{N})$ is closed and $\mathcal{F}$ is a $\mathcal{K}$-coherent family of continuous functions indexed by a P-ideal $\mathcal{J}$. Then one of the following applies:*

(10) *There are $\mathcal{X}_n$ ($n \in \mathbb{N}$) such that $\mathcal{J} = \bigcup_n \mathcal{X}_n$ and each $\mathcal{X}_n$ is $L_1^{\mathcal{K}^6}(\mathbb{N}, n)$-homogeneous.*

(11) *There is an uncountable family $\mathcal{A}$ of pairwise almost disjoint modulo finite sets such that for each $D \in \mathcal{A}$ there is an uncountable strongly $L_0^{\mathcal{K}}(D)$-homogeneous subset of $\mathcal{J}$.*

Proof. This proof is an extension of the proof of [11, Lemma 3.13.5]. Assume (10) fails. By Lemma 1.4, there is an uncountable strongly $L_0^{\mathcal{K}^6}$-homogeneous subset of $\mathcal{J}$. By Lemma 1.5, there is an uncountable $\mathcal{X} \subseteq \mathcal{J}$ that is strongly $L_0^{\mathcal{K}^2}(h)$-homogeneous for some h and forms an increasing ω_1-chain under $\subseteq^*$. Fix $g \colon \mathcal{X} \to \mathcal{P}(\mathbb{N})$ such that

$$\{x, y\} \in L_0^{\mathcal{K}}(D, h(\Delta(g(x), g(y))))$$

for all $\{x, y\} \in [\mathcal{X}]^2$.

Define a forcing notion $\mathcal{P}$ as follows. Conditions are of the form

$$p = (I, k, f, s_\xi, F_\xi : \xi \in I),$$

where

(12) $I \subseteq \omega_1$ is finite,

(13) $k \in \mathbb{N}$,

(14) $f \colon k \to k$ is nonincreasing,

(15) $s_\xi \subseteq k$, and

(16) $F_\xi \subseteq \mathcal{X}$ is finite and such that

$$\{x, y\} \in L_0^{\mathcal{K}}(s_\xi, f(\Delta(g(x), g(y))))$$

for all $\{x, y\} \in [F_\xi]^2$.

The ordering is defined by $p \leq q$ if (write $p = (I^p, k^p, f^p, s_\xi^p, F_\xi^p : \xi \in I^p)$ and similarly for q):

(17) $I^p \supseteq I^q$, $k^p \geq k^q$, $f^p \supseteq f^q$, $s_\xi^p \cap k^q = s_\xi^q$ and $F_\xi^p \supseteq F_\xi^q$ for $\xi \in I^q$.

(18) $s_\xi^p \cap s_\eta^p \subseteq k^q$ for distinct ξ and η in I^q.

Then $\mathcal{P}$ forces that $D_\xi = \bigcup_{p \in G} s_\xi^p$ ($\xi < \omega_1$) are pairwise disjoint modulo finite and if $h = \bigcup_{p \in G} f^p$, then $\mathcal{X}_\xi = \bigcup_{p \in G} F_\xi^p$ is strongly $L_0^{\mathcal{K}}(D_\xi, h)$-homogeneous. The last claim requires that $\lim_n h(n) = \infty$, but the relevant sets are easily seen to be dense in $\mathcal{P}$.

Claim 1.9. *The poset $\mathcal{P}$ is ccc.*

Proof. Let p_α $(\alpha < \omega_1)$ be an uncountable subset of $\mathcal{P}$. We may assume that for some fixed k, $f \colon k \to k$ and all α we have (we will routinely replace the superscript p_α by α) $k^\alpha = k$, $f^\alpha = f$, and that the sets I^α form a Δ-system with root I. Let

$$I = \{\xi_j | j < m\}.$$

A glance at the definition of the ordering of $\mathcal{P}$ shows that p and q are compatible if (let $I = I^p \cap I^q$)

$$(I, k^p, f^p, s^p_\xi, F^p_\xi : \xi \in I) \text{ and } (I, k^q, f^q, s^q_\xi, F^q_\xi : \xi \in I)$$

are compatible. We can therefore assume that $I^\alpha = I$ for all α. We may further assume that for each $j < m$ there is s_j such that $s^\alpha_j = s_j$ for all α, that the sets $\{F^\alpha_j | \alpha < \omega_1\}$ form a Δ-system with root R_j, and that $|F^\alpha_j \setminus R_j| = d_j$ for some fixed d_j.

For each $j \in I$ fix an enumeration

$$F^\alpha_j \setminus R_j = \{A^\alpha_j(i) : i \le d_j\}.$$

By refining further, we can find $k_0 \ge k$, $a_j(i) \subseteq k_0$ and $b_j(i) \subseteq k_0$ for $i \le d_j$ so that

(19) $A^\alpha_j(i) \cap k_0 = a_j(i)$,

(20) $g(A^\alpha_j(i)) \cap k_0 = b_j(i)$.

Increase k_0 and refine further so that for all α and j and $i_1 < i_2 \le d_j$ for $x = A^\alpha_j(i_1)$ and $y = A^\alpha_j(i_2)$ we have

(21) $\Delta(g(x), g(y)) < k_0$,

(22) $[x \upharpoonright k_0] \times [y \upharpoonright k_0] \subseteq L^\mathcal{K}_0(s_j, f(\Delta(g(x), g(y))))$.

By (19)–(22), for each j, all α and β and $i_1 < i_2 \le d_j$ we have that (if $x = A^\alpha_j(i_1)$ and $y = A^\beta_j(i_2)$)

$$\{x, y\} \in L^\mathcal{K}_0(s_j, f(\Delta(g(x), g(y)))).$$

We need to find $\alpha < \beta < \omega_1$ and pairwise disjoint sets $s_j \subseteq [k_0, \infty)$ for $j < m$ such that for all $j < m$ and $i \le d_j$ we have

$$\{A^\alpha_j(i), A^\beta_j(i)\} \in L^\mathcal{K}_0(s_j, k_0).$$

We will recursively find $k_0 < k_1 < \cdots < k_m$ and uncountable $\mathcal{U}_j \subseteq \omega_1$, $\mathcal{W}_j \subseteq \omega_1$ $(j < m)$ such that for every $\alpha \in \mathcal{U}_j$, $\beta \in \mathcal{W}_j$ and $i \le d_j$ we have

$$\{A^\alpha_j(i), A^\beta_j(i)\} \in L^\mathcal{K}_0([k_j, k_{j+1}), k_j).$$

Let $\mathcal{U}_0 = \mathcal{W}_0 = \omega_1$. If we are at stage j of the construction, we can apply Lemma 1.7 d_j times, to find uncountable $\mathcal{U}_{j+1} \subseteq \mathcal{U}_j$, uncountable $\mathcal{W}_{j+1} \subseteq \mathcal{W}_j$, and $k_{j+1} > k_j$ as required.

Pick any $\alpha \in \mathcal{U}_m$ and $\beta \in \mathcal{W}_m$. Define $f^+ \colon k_m \to k_m$ by $f^+(l) = f(l)$ if $l < k$ and $f^+(l) = k$ otherwise. Then

$$q = (I, k_m, f^+, s_j \cup [k_j, k_{j+1}), R_j \cup F^\alpha_j \cup F^\beta_j : j < m)$$

is a joint extension of p_α and p_β.   □

Since $\mathcal{P}$ is ccc, for each $\xi < \omega_1$ there is a condition $p_\xi \in \mathcal{P}$ that forces that $\mathcal{X}_\xi$ is uncountable. Find $p \in \mathcal{P}$ that forces that uncountably many p_ξ belong to G. By applying Martin's Axiom to $\mathcal{P}$ below p and an appropriate family of dense open sets, we can find a filter $G \subseteq \mathcal{P}$ such that for some uncountable $I \subseteq \omega_1$ and all $\xi \in I$ the set $\mathcal{X}_\xi$ is uncountable. We can also assure that $h = \bigcup_{p \in G} f^p$

satisfies $\lim_n h(n) = \infty$. Then each $\mathcal{X}_\xi$ is a strongly $L_0^{\mathcal{K}}(D_\xi, h)$-homogeneous subset of $\mathcal{J}$. □

2. COUNTABLY DETERMINED IDEALS

A map $\phi \colon \mathcal{P}(\mathbb{N}) \to [0, \infty]$ is a *submeasure* if $\phi(\emptyset) = 0$, $\phi(A) \le \phi(B)$ whenever $A \subseteq B$, and $\phi(A \cup B) \le \phi(A) + \phi(B)$ for all A, B. It is *lower semicontinuous* if

$$\phi(A) = \lim_{n \to \infty} \phi(A \cap n)$$

for all A. By a result of Solecki (see [36]), every analytic P-ideal $\mathcal{I}$ is of the form

$$\mathcal{I} = \mathrm{Exh}(\phi) = \Big\{ A \subseteq \mathbb{N} : \lim_{n \to \infty} \phi(A \setminus [1, n)) = 0 \Big\}$$

for some lower semicontinuous submeasure ϕ on $\mathbb{N}$. In particular, every such ideal is $F_{\sigma\delta}$. Moreover, $\mathcal{I}$ is F_σ if and only if ϕ can be chosen to be *exhaustive*, i.e., so that the ideal $\mathrm{Fin}(\phi) = \{A : \phi(A) < \infty\}$ coincides with $\mathrm{Exh}(\phi)$.

A set $\mathcal{K} \subseteq \mathcal{P}(\mathbb{N})$ is *hereditary* if $A \subseteq B \in \mathcal{K}$ implies $A \in \mathcal{K}$. If $\mathcal{K}$ and $\mathcal{L}$ are families of subsets of $\mathbb{N}$, then

$$\mathcal{K} \underline{\cup} \mathcal{L} = \{K \cup L : K \in \mathcal{K} \text{ and } L \in \mathcal{L}\}$$

and

$$\mathcal{K}^k = \{A_1 \cup A_2 \cup \cdots \cup A_k : A_i \in \mathcal{K} \text{ for } i \le k\}.$$

By a result of Mazur ([33, Lemma 1.2(c)]), every F_σ ideal $\mathcal{I}$ is of the form

$$\mathcal{I} = \mathrm{Fin}(\phi) = \{A : \phi(A) < \infty\}$$

for some lower semicontinuous submeasure ϕ.

Definition 2.1. A submeasure ϕ on $\mathbb{N}$ is *nonpathological* if

$$\phi(A) = \sup_\nu \nu(A),$$

where the supremum is taken over all finitely additive measures ν that are pointwise dominated by ϕ. An ideal $\mathcal{I}$ is *nonpathological* if it is of the form $\mathrm{Exh}(\phi)$ or $\mathrm{Fin}(\phi)$ for some lower semicontinuous nonpathological submeasure ϕ.

We should note that our terminology is not standard, as some authors say that ϕ as above is *not weakly pathological*.

Another characterization of F_σ ideals, first stated in [31, Lemma 6.3], is that an ideal $\mathcal{I}$ is F_σ if and only if $\mathcal{I} = \mathcal{K} \underline{\cup} \mathrm{Fin}$ for some closed hereditary set $\mathcal{K}$. Recall that a hereditary set $\mathcal{K}$ is an *approximation* to $\mathcal{I}$ if $\mathcal{K} \underline{\cup} \mathrm{Fin} \supseteq \mathcal{I}$.

Definition 2.2. An ideal $\mathcal{I}$ is *countably determined* by closed (analytic, etc.) approximations if there are closed (analytic, etc.) hereditary sets $\mathcal{K}_n$ ($n \in \mathbb{N}$) such that

(a) $\mathcal{I} = \bigcap_{n=1}^\infty (\mathcal{K}_n \underline{\cup} \mathrm{Fin})$.

If in addition for some $d \in \mathbb{N}$ we have

(b) $\mathcal{I} = \bigcap_{n=1}^\infty (\mathcal{K}_n^d \underline{\cup} \mathrm{Fin})$

we say that $\mathcal{I}$ is *countably d-determined* by closed (analytic, etc.) approximations. If there are closed hereditary $\mathcal{K}_n$ such that (b) holds for all $d \in \mathbb{N}$, we say that $\mathcal{I}$ is *strongly countably determined* by $\mathcal{K}_n$ ($n \in \mathbb{N}$).

In some situations it will be more convenient to consider $\mathcal{K} \underline{\cup} \mathcal{I}$ instead of $\mathcal{K} \underline{\cup} \mathrm{Fin}$.

Lemma 2.3. *If $\mathcal{I}$ is countably d-determined by $\mathcal{K}_m$ ($m \in \mathbb{N}$) and $d \geq 2$, then $\mathcal{I} = \bigcap_{m=1}^{\infty}(\mathcal{K}_m \underline{\cup} \mathcal{I})$.*

Proof. We have $\mathcal{K}_m \underline{\cup} \operatorname{Fin} \subseteq \mathcal{K}_m \underline{\cup} \mathcal{I} \subseteq \mathcal{K}_m^2 \underline{\cup} \operatorname{Fin}$ for all m. $\qquad \square$

The main results of this paper, Theorem 4 and Theorem 5, will be proved for all ideals that are countably 3204-determined by closed approximations. No care is taken to assure the optimality of this constant, in particular because it is possible that all $F_{\sigma\delta}$ ideals are strongly countably determined (or at least countably 3204-determined) by closed approximations.

Lemma 2.4. (1) *Every F_σ ideal is strongly countably determined by closed approximations.*

(2) *Every analytic P-ideal is strongly countably determined by closed approximations.*

(3) *Every ideal countably determined by closed approximations is $F_{\sigma\delta}$.*

Proof. (1) If $\mathcal{K}$ is a closed hereditary set such that $\mathcal{I} = \mathcal{K} \underline{\cup} \operatorname{Fin}$, then let $\mathcal{K}_n = \mathcal{K}$ for all n. We have $\mathcal{K} \subseteq \mathcal{I}$ and therefore $\mathcal{K}^m \subseteq \mathcal{I}$; hence $\mathcal{I} = \mathcal{K}^m \underline{\cup} \operatorname{Fin}$ for all m.

(2) If ϕ is as in Solecki's theorem, let $\mathcal{K}_n = \{A : \phi(A) \leq 1/n\}$.

(3) Obvious. $\qquad \square$

Definition 2.5. Consider the following $F_{\sigma\delta}$ ideals:

$$\operatorname{NWD}(\mathbb{Q}) = \{A \subseteq \mathbb{Q} \cap [0,1] : A \text{ is nowhere dense}\},$$

$$\operatorname{NULL}(\mathbb{Q}) = \{A \subseteq \mathbb{Q} \cap [0,1] : \overline{A} \text{ has Lebesgue measure } 0\},$$

$$\mathcal{Z}_W = \left\{A : \limsup_n \sup_k \frac{|A \cap [k, k+n)|}{n} = 0\right\}.$$

In [16] it was proved that quotients over the ideals $\operatorname{NWD}(\mathbb{Q})$ and $\operatorname{NULL}(\mathbb{Q})$ are both homogeneous, not isomorphic to each other, and not isomorphic to any quotient over an analytic P-ideal. The ideal $\mathcal{Z}_W$ is sometimes called the *Weyl ideal*. Using results of [8] and [16], it can be proved that the quotient over $\mathcal{Z}_W$ is not isomorphic to the quotient over $\operatorname{NWD}(\mathbb{Q})$, $\operatorname{NULL}(\mathbb{Q})$ or any analytic P-ideal.

Lemma 2.6. *The ideals $\operatorname{NULL}(\mathbb{Q})$, $\operatorname{NWD}(\mathbb{Q})$ and $\mathcal{Z}_W$ are strongly countably determined by closed approximations.*

Proof. Enumerate $\mathbb{Q}$ as $\{q_i : i \in \mathbb{N}\}$. Let us prove the case of $\operatorname{NULL}(\mathbb{Q})$ first. Let $\mathcal{F}_n$ be the family of all finite unions of rational intervals of total measure at most 2^{-n}, and enumerate $\mathcal{F}_n$ as $\{U_{ni} : i \in \mathbb{N}\}$. Let $\mathcal{K}_n = \bigcup_{i=1}^{\infty} \mathcal{P}((U_{ni} \cap \mathbb{Q}) \setminus \{q_j : j \leq i\})$. By compactness, for every n every closed null set is covered by some U_{ni}. Therefore every set in $\operatorname{NULL}(\mathbb{Q})$ belongs to $\mathcal{K}_n \underline{\cup} \operatorname{Fin}$ for all n. On the other hand, every set in $\mathcal{K}_n^d$ has measure at most $d2^{-n}$, and therefore $\bigcap_{n=1}^{\infty}(\mathcal{K}_n^d \underline{\cup} \operatorname{Fin}) = \operatorname{NULL}(\mathbb{Q})$ for all d.

The proof that $\operatorname{NWD}(\mathbb{Q})$ has the same property will be more transparent if we consider the dyadic rationals in $\{0,1\}^{\mathbb{N}}$ (which we also denote by $\mathbb{Q}$) instead of $\mathbb{Q}$. If s is a finite partial function from a subset of $\mathbb{N}$ into $\{0,1\}$, then $[s] = \{f \in \{0,1\}^{\mathbb{N}} : f$ extends $s\}$ is a basic open set. Let $\mathcal{S}_n$ be the family of all sets of the form $\bigcup_{i=1}^{n}[s_i]$ for s_i ($i \leq n$) such that $\max \operatorname{dom}(s_i) < \min \operatorname{dom}(s_{i+1})$ for all $i < n$. An easy argument shows that the family $\mathcal{S}_n$ is n-linked, i.e., that an intersection of any n elements of $\mathcal{S}_n$ is a nonempty clopen set (see, e.g., [2, Lemma 2.3.5]). Also, every

nowhere dense subet of $\{0,1\}^{\mathbb{N}}$ is avoided by some element of $\mathcal{S}_n$. Enumerate the basis of $\{0,1\}^{\mathbb{N}}$ as V_n ($n \in \mathbb{N}$) and let

$$\mathcal{F}_n = \{(\mathbb{Q} \cap (V_n \setminus U)) \cup (\mathbb{Q} \setminus V_n) : U \in \mathcal{S}_{2^n}\}.$$

Note that $A \subseteq \mathbb{Q}$ is nowhere dense if and only if for every n we have $A \subseteq W$ for some $W \in \mathcal{F}_n$. Enumerate $\mathcal{F}_n$ as $\{U_{ni} : i \in \mathbb{N}\}$. Let $\mathcal{K}_n = \bigcup_{i=1}^{\infty} \mathcal{P}(U_{ni} \setminus \{q_j : j \leq i\})$. Then $\mathrm{NWD}(\mathbb{Q}) = \bigcap_{n=1}^{\infty} \mathcal{K}_n \underline{\cup} \mathrm{Fin}$. We need to prove that, for every $d \in \mathbb{N}$, if $A \notin \mathrm{NWD}(\mathbb{Q})$, then $A \notin \bigcap_{n=1}^{\infty} \mathcal{K}_n^d \underline{\cup} \mathrm{Fin}$. Fix d and A and find n such that $d < n$ and $A \cap V_n$ is dense in V_n. Then for $W_1, \ldots, W_d \in \mathcal{F}_n$ we have that $V \setminus \bigcup_{i=1}^{d} W_i$ is a nonempty clopen set, thus $A \cap (V \setminus \bigcup_{i=1}^{d} W_i)$ is infinite, and $A \notin \mathcal{K}_n^d \underline{\cup} \mathrm{Fin}$. This completes the proof for $\mathrm{NWD}(\mathbb{Q})$.

Now consider $\mathcal{Z}_W$. Note that

$$\mathcal{Z}_W = \left\{ A : (\forall \varepsilon > 0)(\exists m)(\forall l \geq m)(\forall k) \frac{|A \cap [k, k+l)|}{l} \leq \varepsilon \right\}.$$

The sets $X_{\varepsilon,m} = \{A : (\forall l \geq m)(\forall k)|A \cap [k, k+l)|/l \leq \varepsilon\}$ are closed and hereditary. Sets

$$X_\varepsilon = \{A : A \in X_{\varepsilon,\min(A)}\}$$

are closed and hereditary as well, and $\mathcal{Z}_W = \bigcap_n X_{1/n} \underline{\cup} \mathrm{Fin}$.

Assume $A \in (X_\varepsilon)^d$. Then $A = A_1 \cup A_2 \cup \cdots \cup A_d$ for some k_i and $A_i \in X_{\varepsilon,k_i}$ ($i \leq n$); therefore $A \in X_{d\varepsilon,\max(k_i)}$ and $A \in X_{d\varepsilon} \underline{\cup} \mathrm{Fin}$. Thus we have $(X_\varepsilon)^d \subseteq X_{d\varepsilon} \underline{\cup} \mathrm{Fin}$ and $\mathcal{Z}_W = \bigcap_n (X_{1/n})^d \underline{\cup} \mathrm{Fin}$ for all d. This completes the proof. $\square$

3. Luzin gaps

We are about to introduce 'frozen' gaps (Luzin gaps), as well as those gaps that cannot be frozen by forcing (aloof gaps and unapproachable gaps).

Definition 3.1. Let $\mathcal{I}$ be an ideal and let $\mathcal{K}$ be its analytic approximation. Families $\mathcal{A} = \{A_x : x \in I\}$ and $\mathcal{B} = \{B_x : x \in I\}$ indexed by an uncountable set I form a $\mathcal{K}$-*Luzin gap over* $\mathcal{I}$ if

(1) $\mathcal{A}$ and $\mathcal{B}$ are orthogonal over $\mathcal{I}$ (that is, $A \cap B \in \mathcal{I}$ for all $A \in \mathcal{A}$ and $B \in \mathcal{B}$),
(2) $A_x \cap B_x = \emptyset$ for all $x \in I$, and
(3) $(A_x \cap B_y) \cup (A_y \cap B_x) \notin \mathcal{K} \underline{\cup} \mathcal{K}$, for all distinct x and y in I.

An $\mathcal{I}$-pregap *contains a Luzin gap* if its sides contain a $\mathcal{K}$-Luzin gap for some approximation $\mathcal{K}$ of $\mathcal{I}$.

In the case when $\mathcal{I} = \mathrm{Fin}$ and $\mathcal{K} = \emptyset$, condition (3) reduces to $A_x \cap B_y \neq \emptyset$ or $A_y \cap B_x \neq \emptyset$, the condition originally used by Luzin ([32]), Kunen ([29]) and Todorčević ([38, §8]).

Lemma 3.2. *A Luzin gap cannot be separated by an $\aleph_1$-preserving forcing.*

Proof. We will prove that a Luzin gap is a gap in every forcing extension in which I is uncountable. Note that the conditions from the definition (other than I being uncountable) are absolute. Let $\{A_x : x \in I\}$, $\{B_x : x \in I\}$ be $\mathcal{K}$-Luzin gap over $\mathcal{I}$. Assume that $C \subseteq \mathbb{N}$ is such that $A_x \setminus C \in \mathcal{I}$ and $B_x \cap C \in \mathcal{I}$ for all $x \in I$. Let k be such that the set I' of all $x \in I$ such that

$$((A_x \setminus C) \cup (B_x \cap C)) \setminus k \in \mathcal{K}$$

is uncountable. Find $s \subseteq k$ and $t \subseteq k$ such that the set $J = \{x \in I' : A_x \cap k = s$ and $B_x \cap k = t\}$ is uncountable. Note that $s \cap t = \emptyset$. Then for $x \neq y$ in J we have

$$((A_x \cap B_y) \cup (A_y \cap B_x)) \cap k = s \cap t = \emptyset,$$
$$((A_x \cap B_y) \cup (A_y \cap B_x)) \setminus k \subseteq (A_x \setminus C) \cup (B_y \cap C) \cup (A_y \setminus C) \cup (B_x \cap C)$$
$$\in \mathcal{K} \underline{\cup} \mathcal{K},$$

contradicting the definition of a $\mathcal{K}$-Luzin gap. $\square$

A gap $\mathcal{A}, \mathcal{B}$ over $\mathcal{I}$ is *included in an analytic gap* if there are $\mathcal{A}' \supseteq \mathcal{A}$ and $\mathcal{B}' \supseteq \mathcal{B}$ that form an analytic gap over $\mathcal{I}$.

Definition 3.3. Assume $\mathcal{I}$ is an ideal and $\mathcal{K}$ is its analytic approximation. Two families $\mathcal{A}, \mathcal{B}$ are $\mathcal{K}$-*aloof over* $\mathcal{I}$ if there are closed hereditary sets F and G such that

$$F \underline{\cap} G \subseteq \mathcal{K}$$

while $\mathcal{A} \subseteq F \underline{\cup} \mathcal{I}$ and $\mathcal{B} \subseteq G \underline{\cup} \mathcal{I}$.

We say that $\mathcal{A}$ and $\mathcal{B}$ are *aloof* over $\mathcal{I}$ if there are analytic approximations $\mathcal{K}_m$ ($m \in \mathbb{N}$) such that $\mathcal{I}$ is countably determined by $\mathcal{K}_n$ ($n \in \mathbb{N}$) and $\mathcal{A}$ and $\mathcal{B}$ are $\mathcal{K}_m$-aloof over $\mathcal{I}$ for all $m \in \mathbb{N}$.

Note that in Definition 3.3 and Definition 3.4 below (also in Definition 5.1 below) we consider 'fattenings' of the form $F \underline{\cup} \mathcal{I}$, instead of $F \underline{\cup} \mathrm{Fin}$ as in Definition 2.2.

Definition 3.4. Assume $\mathcal{I}$ is an ideal and $\mathcal{K}$ is a hereditary subset of $\mathcal{P}(\mathbb{N})$. Then two families $\mathcal{A}, \mathcal{B}$ are $\mathcal{K}$-*unapproachable over* $\mathcal{I}$ if there are closed hereditary sets F_n and G_n ($n \in \mathbb{N}$) such that

$$F_n \underline{\cap} G_n \subseteq \mathcal{K}$$

while for every pair $A \in \mathcal{A}$ and $B \in \mathcal{B}$ there is $n \in \mathbb{N}$ such that $A \in F_n \underline{\cup} \mathcal{I}$ and $B \in G_n \underline{\cup} \mathcal{I}$.

We say that $\mathcal{A}$ and $\mathcal{B}$ are *unapproachable* over $\mathcal{I}$ if there are analytic approximations $\mathcal{K}_m$ ($m \in \mathbb{N}$) such that $\mathcal{I}$ is countably determined by $\mathcal{K}_n$ ($n \in \mathbb{N}$) and $\mathcal{A}$ and $\mathcal{B}$ are $\mathcal{K}_m$-unapproachable over $\mathcal{I}$ for all $m \in \mathbb{N}$.

Note that $\mathcal{K} = \{\emptyset\}$ is a closed approximation to Fin.

Lemma 3.5.   (a) *Families $\mathcal{A}$ and $\mathcal{B}$ are $\{\emptyset\}$-unapproachable over Fin if and only if they are countably separated.*

(b) *Families $\mathcal{A}$ and $\mathcal{B}$ are $\{\emptyset\}$-aloof over Fin if and only if they are separated.*

Proof. (a) Let F_n and G_n ($n \in \mathbb{N}$) be as in Definition 3.4. Then $F_n \underline{\cap} G_n = \{\emptyset\}$. Therefore $C_n = \bigcup_n F_n$ and $D_n = \bigcup_n G_n$ are disjoint for every n, and the family C_n ($n \in \mathbb{N}$) separates $\mathcal{A}$ from $\mathcal{B}$.

The proof of (b) is almost identical to the proof of (a). $\square$

Since there are countably separated gaps over Fin (countably separated gaps over Fin are sometimes called Rothberger gaps), the phenomenon of the existence of aloof and unapproachable gaps is not characteristic to more complex analytic ideals. The novelty is in the existence of Hausdorff (i.e., σ-directed) aloof and unapproachable gaps. To the best of my knowledge, it is possible that all unapproachable gaps are aloof.

Lemma 3.6.   (a) *Assume $\mathcal{I}$ is countably determined by approximations $\mathcal{K}_m$ ($m \in \mathbb{N}$). If $\mathcal{A}$ and $\mathcal{B}$ are unapproachable over $\mathcal{I}$ and both $\mathcal{A}/\mathcal{I}$ and $\mathcal{B}/\mathcal{I}$ are σ-directed, then $\mathcal{A}$ and $\mathcal{B}$ are aloof over $\mathcal{I}$.*

   (b) *Assume $\mathcal{K}$ is an approximation to $\mathcal{I}$. If $\mathcal{A}_i$ and $\mathcal{B}$ are $\mathcal{K}$-unapproachable over $\mathcal{I}$ for every i, then $\bigcup_{i=1}^{\infty} \mathcal{A}_i$ and $\mathcal{B}$ are $\mathcal{K}$-unapproachable over $\mathcal{I}$.*

Proof. (a) For each m let $F_{m,n}$, $G_{m,n}$ ($n \in \mathbb{N}$) be closed hereditary sets such that $F_{m,n} \underline{\cap} G_{m,n} \subseteq \mathcal{K}_m$ and for all $A \in \mathcal{A}$ and $B \in \mathcal{B}$ there is $n \in \mathbb{N}$ such that $A \in F_{m,n} \underline{\cup} \mathcal{I}$ and $B \in G_{m,n} \underline{\cup} \mathcal{I}$.

   Since $\mathcal{A}$ is σ-directed modulo $\mathcal{I}$, for each m there is an $n = f(m)$ such that $F_{m,f(m)}$ is cofinal in $\mathcal{A}/\mathcal{I}$ (see, e.g., [11, Lemma 2.2.2]). Let $F_m = F_{m,f(m)}$. Then $\mathcal{A} \subseteq F_m \underline{\cup} \mathcal{I}$. Similarly, the family $\mathcal{B}'$ generated by $\mathcal{B}$ and $\mathcal{I}$ is σ-directed modulo Fin, and we chose $G_m = G_{m,g(m)}$ that is cofinal in $\mathcal{B}/\mathcal{I}$, so that we have $\mathcal{B} \subseteq G_m \underline{\cup} \mathcal{I}$. Therefore F_m, G_m ($m \in \mathbb{N}$) are as required.

   (b) If $F_{m,n}^i$ and $G_{m,n}^i$ ($m, n \in \mathbb{N}$) witness that $\mathcal{A}_i$ and $\mathcal{B}_i$ are $\mathcal{K}$-unapproachable, then

$$F_{m,n} = \bigcup_{i=1}^{\infty} (F_{m,n}^i \cap \mathcal{P}([i,\infty))),$$

$$G_{m,n} = \bigcup_{i=1}^{\infty} (G_{m,n}^i \cap \mathcal{P}([i,\infty))).$$

are compact, hereditary, and witness that $\mathcal{A}$ and $\mathcal{B}$ are $\mathcal{K}$-unapproachable. $\square$

4. A dichotomy for gaps

The variant of the following lemma for $\mathcal{I} = \text{Fin}$ (due to Todorčević) is one of the earliest applications of OCA.

Lemma 4.1. *Let $\mathcal{I}$ be an ideal and let $\mathcal{K}$ be its closed approximation. To a pre-gap $\mathcal{A}, \mathcal{B}$ in its quotient we can associate a separable metric space $\mathcal{X}$ and its open partition $[\mathcal{X}]^2 = K_0^{\mathcal{K}} \cup K_1^{\mathcal{K}}$ so that*

   (a) *if $\mathcal{X}$ is σ-$K_1^{\mathcal{K}}$-homogeneous then $\mathcal{A}, \mathcal{B}$ are $\mathcal{K}^2$-unapproachable over $\mathcal{I}$, and*

   (b) *an uncountable $K_0^{\mathcal{K}}$-homogeneous subset of $\mathcal{X}$ is a $\mathcal{K}$-Luzin gap over $\mathcal{I}$.*

Proof. We may assume that $\mathcal{A}$ and $\mathcal{B}$ are hereditary. Then

$$\mathcal{X} = \mathcal{A} \otimes \mathcal{B} = \{(A, B) : A \in \mathcal{A}, B \in \mathcal{B} \text{ and } A \cap B = \emptyset\}$$

has the property that for all $A \in \mathcal{A}$ and $B \in \mathcal{B}$ we have $(A', B') \in \mathcal{X}$ for some A' and B' such that $A \Delta A' \in \mathcal{I}$ and $B \Delta B' \in \mathcal{I}$. For simplicity of notation, we will denote elements of $\mathcal{X}$ by $p = (A_p, B_p)$. Define a partition $[\mathcal{X}]^2 = K_0^{\mathcal{K}} \cup K_1^{\mathcal{K}}$ by

$$\{p, q\} \in K_0^{\mathcal{K}} \Leftrightarrow ((A_p \cap B_q) \cup (A_q \cap B_p)) \notin \mathcal{K} \underline{\cup} \mathcal{K}.$$

Since $\mathcal{K}$ is closed, this partition is open in the separable metric topology on X induced from $\mathcal{P}(\mathbb{N}) \times \mathcal{P}(\mathbb{N})$.

   If $I \subseteq X$ is uncountable and $K_0^{\mathcal{K}}$-homogeneous, then it is a $\mathcal{K}$-Luzin subgap of $\mathcal{A}, \mathcal{B}$. If there is no such I, by OCA we can write $\mathcal{X} = \bigcup_{n=1}^{\infty} \mathcal{X}_n$, where $[\mathcal{X}_n]^2 \cap K_0^{\mathcal{K}} = \emptyset$ for all n. Let

$$F_n = \overline{\{A \in \mathcal{A} : (\exists B \in \mathcal{B})(A, B) \in \mathcal{X}_n\}},$$

$$G_n = \overline{\{B \in \mathcal{B} : (\exists A \in \mathcal{A})(A, B) \in \mathcal{X}_n\}}.$$

Since $\mathcal{X}$ is covered by $\bigcup_n \mathcal{X}_n$, for every pair $A \in \mathcal{A}$ and $B \in \mathcal{B}$ there is n such that $A \in F_n \underline{\cup} \mathcal{I}$ and $B \in G_n \underline{\cup} \mathcal{I}$.

We need to check the orthogonality requirement on F_n and G_n. Pick a finite $d \in F_n \cap G_n$ for some n. Then for some $p, q \in \mathcal{X}_n$ we have $d \subseteq A_p$ and $d \subseteq B_q$; thus $d \in \mathcal{K} \underline{\cup} \mathcal{K}$ by the homogeneity of $\mathcal{X}_n$. Therefore $\mathcal{A}$ and $\mathcal{B}$ are $\mathcal{K}^2$-unapproachable. $\quad\square$

Theorem 4.2 (OCA). *Let $\mathcal{I}$ be an ideal countably 2-determined by its closed approximations $\mathcal{K}_m$ $(m \in \mathbb{N})$.*

(a) *A pair of $\mathcal{I}$-orthogonal families $\mathcal{A}, \mathcal{B}$ either is unapproachable or it contains a Luzin gap.*

(b) *If $\mathcal{A}, \mathcal{B}$ moreover form a Hausdorff gap, then $\mathcal{A}, \mathcal{B}$ either is aloof or it contains a Hausdorff Luzin gap.*

Proof. Clause (a) follows immediately from Lemma 4.1 applied to each $\mathcal{K}_m$.

(b) By (a) we may assume that $\mathcal{A}, \mathcal{B}$ are either unapproachable over $\mathcal{I}$ or they include a Luzin gap over $\mathcal{I}$. If $\mathcal{A}$ and $\mathcal{B}$ are unapproachable over $\mathcal{I}$, then they are aloof over $\mathcal{I}$ by (a) of Lemma 3.6.

Now assume A_ξ, B_ξ $(\xi \in I)$ form a Luzin subgap of a Hausdorff gap $\mathcal{A}, \mathcal{B}$. We may assume $I = \omega_1$. Recursively find $A'_\xi \in \mathcal{A}$ such that for all $\eta < \xi$ we have $A_\xi \setminus A'_\xi \in \mathcal{I}$ and $A'_\eta \setminus A'_\xi \in \mathcal{I}$. Recursively find $B'_\xi \in \mathcal{B}$ such that for all $\eta < \xi$ we have $B_\xi \setminus B'_\xi \in \mathcal{I}$ and $B'_\eta \setminus B'_\xi \in \mathcal{I}$. Then $A''_\xi = A_\xi \cup (A'_\xi \setminus (B_\xi \cup B'_\xi))$ and $B''_\xi = B_\xi \cup (B'_\xi \setminus A_\xi)$ (for $\xi < \omega_1$) form a Hausdorf Luzin gap in $\mathcal{P}(\mathbb{N})/\mathcal{I}$. $\quad\square$

Lemma 4.3. *If $\mathcal{I}$ is an ideal and $\mathcal{K}$ is its closed approximation and $\mathcal{A}, \mathcal{B}$ is a pregap in $\mathcal{P}(\mathbb{N})/\mathcal{I}$ that is not $\mathcal{K}^2$-unapproachable over $\mathcal{I}$, then a proper poset adds a $\mathcal{K}$-Luzin subgap to it.*

Proof. By Lemma 4.1, if $\mathcal{A}, \mathcal{B}$ is not $\mathcal{K}^2$-unapproachable over $\mathcal{I}$, then $\mathcal{X} = \mathcal{A} \otimes \mathcal{B}$ is not σ-$K_1^{\mathcal{K}}$-homogeneous. Force CH without adding reals. Then by [38, Theorem 4.4] there is a ccc poset that adds an uncountable $K_0^{\mathcal{K}}$-homogeneous subset to $\mathcal{X}$, and by Lemma 4.1 this set is a Luzin gap. $\quad\square$

The following is an immediate consequence of Lemma 4.3.

Theorem 4.4. *Assume $\mathcal{I}$ is countably 2-determined by its closed approximations and $\mathcal{A}, \mathcal{B}$ is a gap in $\mathcal{P}(\mathbb{N})/\mathcal{I}$. Then either $\mathcal{A}, \mathcal{B}$ is unapproachable over $\mathcal{I}$ or a proper poset adds a Luzin subgap to $\mathcal{A}, \mathcal{B}$.*

If $\mathcal{A}, \mathcal{B}$ is moreover a Hausdorff gap, then either it is aloof or a proper poset adds a Hausdorff Luzin subgap to it. $\quad\square$

5. Approximations to liftings. I

Let us recall some notions and results about liftings from [11, Chapter 1]. Let $\mathcal{I}$ and $\mathcal{J}$ be ideals on $\mathbb{N}$, and let $\Phi \colon \mathcal{P}(\mathbb{N})/\mathcal{I} \to \mathcal{P}(\mathbb{N})/\mathcal{J}$ be a homomorphism. A map $\Phi_* \colon \mathcal{P}(\mathbb{N}) \to \mathcal{P}(\mathbb{N})$ which induces Φ by making the diagram

$$
\begin{array}{ccc}
\mathcal{P}(\mathbb{N}) & \xrightarrow{\;\Phi_*\;} & \mathcal{P}(\mathbb{N}) \\
\downarrow{\scriptstyle \pi_\mathcal{I}} & & \downarrow{\scriptstyle \pi_\mathcal{J}} \\
\mathcal{P}(\mathbb{N})/\mathcal{I} & \xrightarrow{\;\Phi\;} & \mathcal{P}(\mathbb{N})/\mathcal{J}
\end{array}
$$

commute is a *lifting* of Φ. (Note that we **do not** require the lifting to be a homomorphism itself!) When we talk about continuous, Borel or Baire-measurable liftings, we are referring to the compact metric topology obtained by identifying $\mathcal{P}(\mathbb{N})$ with the Cantor set. If $\mathcal{X} \subseteq \mathcal{P}(\mathbb{N})$ and $f(A)\Delta\Phi_*(A) \in \mathcal{I}$ for all $A \in \mathcal{X}$, we say that f is a *lifting of Φ on $\mathcal{X}$*. Note that f is a Baire-measurable lifting of Φ on $\mathcal{X}$ means that f is a Baire-measurable function whose domain is a Baire-measurable set such that its restriction to $\mathcal{X}$ is a lifting of Φ. It is not required that $\mathcal{X}$ be Baire-measurable.

If a graph of a lifting of an automorphism of $\mathcal{P}(\mathbb{N})/\operatorname{Fin}$ is covered by countably many graphs of Baire-measurable functions, then the automorphism has a continuous lifting ([44]). This is no longer true for homomorphisms; for example, the homomorphism $\Phi\colon \mathcal{P}(\mathbb{N})/\operatorname{Fin} \to \mathcal{P}(\mathbb{N})/\operatorname{Fin}$ from [11, Example 3.2.1] has a lifting whose graph is covered by graphs of two continuous functions, but Φ does not have a continuous lifting. In this section we will deal with homomorphisms with a lifting that can be approximated by countably many Baire-measurable functions.

Definition 5.1. If $\mathcal{K}$ is a hereditary subset of $\mathcal{P}(\mathbb{N})$ and $f, g\colon \mathcal{P}(\mathbb{N}) \to \mathcal{P}(\mathbb{N})$ and $\mathcal{X} \subseteq \mathcal{P}(\mathbb{N})$, we say that g is a *$\mathcal{K}$-approximation to f on $\mathcal{X}$* if

$$f(A)\Delta g(A) \in \mathcal{K}\underline{\cup}\mathcal{I}$$

for every $A \in \mathcal{X}$. If $\mathcal{X} = \mathcal{P}(\mathbb{N})$, then we just say that g is a $\mathcal{K}$-approximation to f. If $\Phi\colon \mathcal{P}(\mathbb{N}) \to \mathcal{P}(\mathbb{N})/\mathcal{I}$ is a homomorphism and $\mathcal{K}$ is an approximation to $\mathcal{I}$, we say that f is a *$\mathcal{K}$-approximation to Φ* if it is a $\mathcal{K}$-approximation to some (equivalently, every) lifting of Φ.

Assume that $\mathcal{I}$ is an ideal and $\mathcal{K}$ is its approximation. To a homomorphism $\Phi\colon \mathcal{P}(\mathbb{N}) \to \mathcal{P}(\mathbb{N})/\mathcal{I}$ we associate the following ideals (cf. [11, p. 103]):

$$\mathcal{J}_{\mathrm{cont}}^{\mathcal{K}}(\Phi) = \{A : \Phi \restriction \mathcal{P}(A) \text{ has a continuous } \mathcal{K}\text{-approximation}\},$$

$$\mathcal{J}_{\sigma}^{\mathcal{K}}(\Phi) = \{A : \Phi \restriction \mathcal{P}(A) \text{ has a } \mathcal{K}\text{-approximation whose graph is covered}$$
$$\text{by graphs of countably many Borel-measurable functions}\},$$

$$\mathcal{J}_{\sigma-}^{\mathcal{K}}(\Phi) = \{A : \Phi \restriction \mathcal{P}(A) \text{ has a } \mathcal{K}\text{-approximation whose graph is covered}$$
$$\text{by graphs of countably many Baire-measurable functions}\}.$$

We write $\mathcal{J}_{\mathrm{cont}}(\Phi) = \mathcal{J}_{\mathrm{cont}}^{\mathcal{I}}(\Phi)$, $\mathcal{J}_{\sigma}(\Phi) = \mathcal{J}_{\sigma}^{\mathcal{I}}(\Phi)$ and $\mathcal{J}_{\sigma-}(\Phi) = \mathcal{J}_{\sigma-}^{\mathcal{I}}(\Phi)$. In all lemmas below we assume that $\Phi\colon \mathcal{P}(\mathbb{N}) \to \mathcal{P}(\mathbb{N})/\mathcal{I}$ is a homomorphism.

Lemma 5.2. *If $\mathcal{K}$ is an analytic approximation to $\mathcal{I}$, then $\mathcal{J}_{\sigma-}^{\mathcal{K}}(\Phi) \subseteq \mathcal{J}_{\sigma}^{\mathcal{K}^2}(\Phi)$.*

Proof. Assume $A \in \mathcal{J}_{\sigma-}^{\mathcal{K}}(\Phi)$ and f_n $(n \in \mathbb{N})$ are Baire-measurable functions witnessing this. Let G be a relatively comeager subset of $\mathcal{P}(A)$ such that each f_n is continuous on G. Find a disjoint partition $A = A_0 \cup A_1$ and $S_0 \subseteq A_0$, $S_1 \subseteq A_1$ so that $X \cup S_{1-i} \in G$ for all $X \subseteq A_i$, for $i < 2$. Let $C = \Phi_*(A_0)$. For $m, n \in \mathbb{N}$ the function

$$g_{m,n}(B) = ((f_m((B \cap A_0) \cup S_1)) \cap C) \cup ((f_n((B \cap A_1) \cup S_0)) \setminus C)$$

is continuous. Moreover, if $B \subseteq A$ and $m(0), m(1)$ are such that

$$f_{m(i)}(B \cap A_i)\Delta\Phi_*(B \cap A_i) \in \mathcal{K}\underline{\cup}\mathcal{I}$$

for $i < 2$, then $g_{m,n}(B)\Delta\Phi_*(B) \in \mathcal{K}^2\underline{\cup}\mathcal{I}$. Therefore $g_{m,n}$ $(m, n \in \mathbb{N})$ witness that $A \in \mathcal{J}_{\sigma}^{\mathcal{K}^2}(\Phi)$. $\square$

Lemma 5.3. *Assume $\mathcal{I}$ is countably 2-determined by its analytic approximations $\mathcal{K}_m$ $(m \in \mathbb{N})$. Then*

$$\bigcap_{m=1}^{\infty} \mathcal{J}_{\mathrm{cont}}^{\mathcal{K}_m}(\Phi) = \mathcal{J}_{\mathrm{cont}}(\Phi).$$

Proof. We only need to prove the direct inclusion. Pick $C \in \bigcap_{m=1}^{\infty} \mathcal{J}_{\mathrm{cont}}^{\mathcal{K}_m}(\Phi)$. For each $m \in \mathbb{N}$ let f_m be a continuous $\mathcal{K}_m$-approximation to Φ_* on $\mathcal{P}(C)$. The set

$$\mathcal{X} = \{(A, B) : A \in \mathcal{P}(C) \text{ and } (\forall m) f_m(A) \Delta B \in \mathcal{K}_m \underline{\sqcup} \mathcal{I}\}$$

is analytic. By the Jankov–von Neumann uniformization theorem ([28, Theorem 18.1]) this set can be uniformized by a Baire-measurable function f. By Lemma 2.3 we have $\bigcap_{n=1}^{\infty} \mathcal{K}_n \underline{\sqcup} \mathcal{I} = \mathcal{I}$, so $f(A) \Delta \Phi_*(A) \in \mathcal{I}$ for all $A \in \mathcal{P}(C)$. Therefore the restriction of Φ to $\mathcal{P}(C)$ has a Baire-measurable lifting f. By Lemma 6.3, it has a continuous lifting as well. $\square$

Two subsets of $\mathbb{N}$ are *almost disjoint* if they are disjoint modulo Fin. We will follow the established terminology and say that a family of subsets of $\mathbb{N}$ that are pairwise almost disjoint is an *almost disjoint family*.

Lemma 5.4. *Assume $\mathcal{K}$ is an analytic approximation to $\mathcal{I}$. If A_n $(n \in \mathbb{N})$ are pairwise almost disjoint and $\bigcup_{n=1}^{\infty} A_n \in \mathcal{J}_{\sigma}^{\mathcal{K}}(\Phi)$, then for all but finitely many n we have $A_n \in \mathcal{J}_{\mathrm{cont}}^{\mathcal{K}^8}(\Phi)$.*

Proof. The case when $\mathcal{I}$ is an analytic P-ideal $\mathrm{Exh}(\phi)$ for a lower semicontinuous submeasure ϕ and $\mathcal{K} = \{A : \phi(A) \le 2^{n-3}\}$ was proved in [11, Lemma 3.12.3]. The proof of the present lemma is identical. (A more elegant proof, using measure in place of category, can be found in [18, Lemma 3C].) $\square$

An almost disjoint family $\mathcal{A}$ is *linear* if it can be well-ordered so that every initial segment is separated from its complement in $\mathcal{A}$ over Fin. Note that if f_{ξ} $(\xi < \omega_1)$ is a strictly $<^*$-increasing chain of functions in $\mathbb{N}^{\mathbb{N}}$, then the almost disjoint family

$$A_{\xi} = \{(m, n) : f_{\xi}(m) \le n < f_{\xi+1}(m)\}$$

of subsets of $\mathbb{N}^2$ is linear.

An ideal $\mathcal{J}$ is *ccc over* Fin if every almost disjoint family of $\mathcal{J}$-positive sets is countable ([11, Definition 3.3.1]). An ideal $\mathcal{J}$ is *linearly ccc over* Fin if every linear almost disjoint family of $\mathcal{J}$-positive sets is countable. I do not know whether '$\mathcal{I}$ is linearly ccc over Fin' is equivalent to '$\mathcal{I}$ is ccc over Fin.' I also do not know whether Lemma 5.7 holds for ccc over Fin ideals.

Lemma 5.5. *For an ideal $\mathcal{J} \subseteq$ Fin each of the following conditions implies the next.*

(1) *$\mathcal{J}$ is ccc over* Fin,
(2) *$\mathcal{J}$ is linearly ccc over* Fin,
(3) *$\mathcal{J}$ is nonmeager.*

Proof. Only (2) implies (3) requires a proof. Assume $\mathcal{J}$ is meager. By a well-known characterization of nonmeager ideals (Jalali–Naini [21] and Talagrand [37]) there is a family u_{mn} $(m, n \in \mathbb{N})$ of finite pairwise disjoint subsets of $\mathbb{N}$ such that every infinite union of these sets is $\mathcal{J}$-positive. Let f_{α} $(\alpha < \omega_1)$ be functions in $\mathbb{N}^{\mathbb{N}}$ such that for all $\alpha < \beta$ the set $\{n : f_{\alpha}(n) \ge f_{\beta}(n)\}$ is finite. Then $A_{\alpha} = \bigcup\{u_{mn} : f_{\alpha}(m) \le n \le f_{\alpha+1}(m)\}$ $(\alpha < \omega_1)$ is an almost disjoint family of $\mathcal{J}$-positive sets.

Since the sets $B_\alpha = \bigcup\{u_{mn} : n \le f_{\alpha+1}(m)\}$ separate initial segments from end segments, $\mathcal{J}$ is not linearly ccc over Fin. $\qquad\square$

Lemma 5.6. *The ideal $\mathcal{J}_{\mathrm{cont}}(\Phi)$ is (linearly) ccc over* Fin *if and only if the ideal $\mathcal{J}_{\mathrm{cont}}^{\mathcal{K}_m}(\Phi)$ is (linearly) ccc over* Fin *for every m.*

Proof. Since $\mathcal{J}_{\mathrm{cont}}(\Phi) \subseteq \mathcal{J}_{\mathrm{cont}}^{\mathcal{K}_m}(\Phi)$ for all m, only the converse direction requires a proof. Assume $\mathcal{J}_{\mathrm{cont}}^{\mathcal{K}_m}(\Phi)$ is (linearly) ccc over Fin for every m. If A_α $(\alpha < \omega_1)$ is a (linear) almost disjoint family, then for all but countably many α we have $A_\alpha \in \bigcap_{m=1}^{\infty} \mathcal{J}_{\mathrm{cont}}^{\mathcal{K}_m}(\Phi)$ for all m. The conclusion now follows by Lemma 5.3. $\qquad\square$

Lemma 5.7. *If $\mathcal{K}$ is an analytic approximation to $\mathcal{I}$ and $\mathcal{J}_\sigma^{\mathcal{K}}(\Phi)$ is linearly ccc over* Fin*, then $\mathcal{J}_{\mathrm{cont}}^{\mathcal{K}^8}(\Phi)$ is linearly ccc over* Fin*.*

Proof. Assume $\mathcal{J}_\sigma^{\mathcal{K}}(\Phi)$ is linearly ccc over Fin and $\mathcal{J}_{\mathrm{cont}}^{\mathcal{K}^8}(\Phi)$ is not linearly ccc over Fin. Let A_α $(\alpha < \omega_1)$ be a linear almost disjoint family of $\mathcal{J}_{\mathrm{cont}}^{\mathcal{K}^8}(\Phi)$-positive sets. Using the linearity of this family, we can recursively find sets B_α $(\alpha < \omega_1)$ such that for all $\alpha, \beta < \omega_1$ and $n \in \omega$, we have

  (1) $A_{\alpha\omega+n} \subseteq^* B_\alpha$,
  (2) $A_{\alpha\omega+n} \cap B_\beta$ is finite, if $\alpha \ne \beta$, and
  (3) $B_\alpha \cap B_\beta$ is finite, if $\alpha \ne \beta$.

Since $\mathcal{J}_\sigma^{\mathcal{K}}(\Phi)$ is linearly ccc over Fin, some B_α belongs to this ideal. By Lemma 5.4, some $A_{\alpha\omega+n}$ is in $\mathcal{J}_{\mathrm{cont}}^{\mathcal{K}^8}(\Phi)$. $\qquad\square$

6. Almost liftings

We say that a map $F \colon \mathcal{P}(\mathbb{N}) \to \mathcal{P}(\mathbb{N})$ is an *almost lifting* of some $\Phi \colon \mathcal{P}(\mathbb{N}) \to \mathcal{P}(\mathbb{N})/\mathcal{I}$ if the set $\{A : F(A)\Delta\Phi_*(A) \in \mathcal{I}\}$ includes a nonmeager ideal $\mathcal{J}$. The reader should be warned that this differs from the definition given in [11] where it was required that $\mathcal{J}$ is ccc over Fin. Since all ccc over Fin ideals are nonmeager, results about almost liftings in the present sense are stronger than the results about almost liftings from [11].

If $\Phi \colon \mathcal{P}(\mathbb{N}) \to \mathcal{P}(\mathbb{N})/\mathcal{I}$ and $A \subseteq \mathbb{N}$, then Φ^A is the map from $\mathcal{P}(\mathbb{N})$ into $\mathcal{P}(\mathbb{N})/\mathcal{I}$ whose lifting is

$$B \mapsto \Phi_*(B) \cap A,$$

where Φ_* is any lifting of A.

Lemma 6.1. *If an ideal $\mathcal{I}$ is strongly countably determined by closed approximations, $\Phi \colon \mathcal{P}(\mathbb{N}) \to \mathcal{P}(\mathbb{N})/\mathcal{I}$ is a homomorphism, and $\mathcal{J}$ is a nonmeager ideal, then the following are equivalent.*

  (a) *There is $A \subseteq \mathbb{N}$ such that Φ^A has a continuous lifting and $\ker(\Phi^{\mathbb{N}\setminus A}) \supseteq \mathcal{J}$.*
  (b) *Φ has a continuous almost lifting on $\mathcal{J}$.*

Proof. Assume (a). If F is a continuous lifting of Φ^A, then it is a lifting of Φ on $\ker(\Phi^A) \supseteq \mathcal{J}$, and hence (b) follows.

The converse direction was proved in [11, Lemma 3.11.6] in the case when $\mathcal{I}$ is an analytic P-ideal. The proof given there gives the present lemma once the (n_i, n_{i+1})-2^{-i}-stabilizers are replaced with (n_i, n_{i+1})-$\mathcal{K}_i$-stabilizers, defined in the natural manner (see [11, p. 95]). Here $\mathcal{K}_i$ $(i \in \mathbb{N})$ are the closed approximations to $\mathcal{I}$ that strongly generate it. $\qquad\square$

Lemma 6.2. *If $\mathcal{I}_2$ is strongly countably determined by closed approximations, then an isomorphism $\Phi\colon \mathcal{P}(\mathbb{N})/\mathcal{I}_1 \to \mathcal{P}(\mathbb{N})/\mathcal{I}_2$ has a continuous almost lifting if and only if it has a continuous lifting.*

Proof. We only need to prove the direct implication, so assume that an isomorphism $\Phi\colon \mathcal{P}(\mathbb{N})/\mathcal{I}_1 \to \mathcal{P}(\mathbb{N})/\mathcal{I}_2$ has a continuous almost lifting and that $\mathcal{I}_2$ is strongly countably determined by closed approximations. By Lemma 6.1 there is $A \subseteq \mathbb{N}$ such that $\ker(\Phi^{\mathbb{N}\setminus A})$ includes a nonmeager ideal and Φ^A has a continuous lifting. Let $B = \Phi^{-1}(\mathbb{N} \setminus A)$. Then $\ker(\Phi) \cap \mathcal{P}(B) = \ker(\Phi^{\mathbb{N}\setminus A}) \cap \mathcal{P}(B)$ is a nonmeager analytic ideal including Fin. Such an ideal is improper ([21, 37]); therefore $B \in \mathcal{I}_1$, $\mathbb{N} \setminus A \in \mathcal{I}_2$, and the continuous lifting of Φ^A is a lifting of Φ as well. $\square$

The case when $\mathcal{J} = \mathcal{P}(\mathbb{N})$ of the following lemma is well-known ([43, p. 132], [41, Theorem 3]), and it shows that assuming Φ to have a continuous almost lifting is as general as assuming it to have an almost lifting 'definable' in some reasonable sense.

Lemma 6.3. *If a homomorphism $\Phi\colon \mathcal{P}(\mathbb{N}) \to \mathcal{P}(\mathbb{N})/\mathcal{I}$ has a Baire-measurable lifting on a nonmeager ideal $\mathcal{J}$, then it also has a continuous lifting on $\mathcal{J}$.* $\square$

Proof. The argument is a simpler version of an argument from the proof of [11, Lemma 3.11.4]. Let f be a lifting that is Baire-measurable on $\mathcal{X}$. We may assume $\mathrm{dom}(f) = \mathcal{P}(\mathbb{N})$. Find a dense G_δ set G on which f is continuous, and find an increasing sequence n_i ($i \in \mathbb{N}$) and $s_i \subseteq [n_i, n_{i+1})$ such that $\{X : (\exists^\infty i)X \cap [n_i, n_{i+1}) = s_i\} \subseteq G$. Since $\mathcal{J}$ is nonmeager, by [21, 37] there is an infinite set $C \subseteq \mathbb{N}$ such that $\bigcup_{i \in C} s_i \in \mathcal{J}$. Let $m(i)$ ($i \in \mathbb{N}$) be the increasing enumeration of C. Let $A_0 = \bigcup_i [n_{m(2i)}, n_{m(2i+1)})$, $A_1 = \mathbb{N} \setminus A_0$, $S_0 = \bigcup_i s_{m(2i)}$, $S_1 = \bigcup_i s_{m(2i+1)}$. Then

$$g(X) = (f((X \cap A_0) \cup S_1) \cap \Phi_*(A_0)) \cup (f(X \cap A_1) \cup S_0) \cap \Phi_*(A_1))$$

is a continuous map that is a lifting of Φ on $\mathcal{J}$. $\square$

7. APPROXIMATIONS TO LIFTINGS. II

If $\Phi\colon \mathcal{P}(\mathbb{N}) \to \mathcal{P}(\mathbb{N})/\mathcal{I}$ is a homomorphism and $\mathcal{K}$ is an approximation to $\mathcal{I}$, we define the following ideals.

$$\mathcal{J}_2(\Phi) = \{a \subseteq \mathbb{N} : \ \Phi \restriction \mathcal{P}(a) \text{ has a continuous almost lifting, } \Psi_a\},$$

$$\mathcal{J}_2^{\mathcal{K}}(\Phi) = \{a \subseteq \mathbb{N} : \ \Phi \restriction \mathcal{P}(a) \text{ has a continuous } \mathcal{K}\text{-approximation}$$
$$\text{on a nonmeager (relative to } \mathcal{P}(a)) \text{ ideal}\}.$$

Clearly $\mathcal{J}_2(\Phi) \supseteq \mathcal{J}_{\mathrm{cont}}(\Phi)$. Let us redo Lemma 5.3.

Lemma 7.1. *Assume $\mathcal{I}$ is countably determined by its analytic approximations $\mathcal{K}_m$ ($m \in \mathbb{N}$). Then*

$$\bigcap_{m=1}^{\infty} \mathcal{J}_2^{\mathcal{K}_m}(\Phi) = \mathcal{J}_2(\Phi).$$

Proof. Since $\mathcal{J}_2(\Phi) \subseteq \mathcal{J}_2^{\mathcal{K}_m}(\Phi)$ for all m, we need only to prove the direct inclusion. Pick $C \in \bigcap_{m=1}^{\infty} \mathcal{J}_2^{\mathcal{K}_m}(\Phi)$. For each $m \in \mathbb{N}$ let f_m be a continuous $\mathcal{K}_m$-approximation to Φ on $\mathcal{P}(C) \cap \mathcal{L}_m$, for a ccc over Fin ideal $\mathcal{L}_m$ on C. The set

$$\mathcal{X} = \{(A, B) : A \in \mathcal{P}(C) \text{ and } (\forall m)f_m(A)\Delta B \in \mathcal{K}_m \underline{\sqcup} \mathcal{I}\}$$

is analytic. By the Jankov–von Neumann uniformization theorem ([28, Theorem 18.1]) this set can be uniformized by a Baire-measurable function f. Then $f(A)\Delta\Phi_*(A) \in \mathcal{I}$ for all $A \in \bigcap_m \mathcal{L}_m$. The intersection of countably many nonmeager ideals, each of which includes Fin, is nonmeager. This is an easy consequence of the Jalali-Naini–Talagrand characterization of nonmeager ideals ([21], [37]). Therefore $\bigcap_m \mathcal{L}_m$ is nonmeager, and the restriction of Φ to $\mathcal{P}(C)$ has a Baire-measurable almost lifting. By Lemma 6.3, it has a continuous almost lifting as well. $\square$

The following is an improved version of [11, Claim 3 on page 108], and it was essentially proved in [18, Lemma 3F].

Lemma 7.2. *Assume that $\mathcal{F}$ is a family of pairs (H, A) such that $A \subseteq \mathbb{N}$ and $H\colon \mathcal{P}(A) \to \mathcal{P}(\mathbb{N})$ is a continuous map, and that for some closed $\mathcal{K} \subseteq \mathcal{P}(\mathbb{N})$ and all (H, A) and (G, B) in $\mathcal{F}$ we have*

$$H(s)\Delta G(s) \in \mathcal{K}$$

for all $s \subseteq A \cap B$. Then there is a Baire-measurable function $F\colon \mathcal{P}(\mathbb{N}) \to \mathcal{P}(\mathbb{N})$ such that for all $(H, A) \in \mathcal{F}$ and $B \subseteq A$ we have $H(B)\Delta F(B) \in \mathcal{K}$.

Proof. Let $\mathcal{F}_0 \subseteq \mathcal{F}$ be countable and such that for every $(H, A) \in \mathcal{F}$ and $k \in \mathbb{N}$ there is $(G, B) \in \mathcal{F}_0$ satisfying

(23) $A \cap k = B \cap k$, and
(24) $H(s) \cap k = G(s) \cap k$ for all $s \subseteq A \cap k$.

Let $\mathcal{H} \subseteq \mathcal{P}(\mathbb{N})$ be the set of all pairs $(C, D) \in \mathcal{P}(\mathbb{N})^2$ such that

(25) $(\forall k \in \mathbb{N})(\exists (H, A) \in \mathcal{F}_0)(\exists k' \geq k)$

$$C \cap k' \subseteq A \text{ and } H(C \cap k') \cap k = D \cap k.$$

Then $\mathcal{H}$ is Borel, so by the Jankov–von Neumann uniformization theorem there is a Baire-measurable $F\colon \mathcal{P}(\mathbb{N}) \to \mathcal{P}(\mathbb{N})$ such that $(C, F(C)) \in \mathcal{H}$ for every C such that there is D satisfying $(C, D) \in \mathcal{H}$.

Claim 7.3. *If $(H, A) \in \mathcal{F}$ and $C \subseteq A$, then $(C, H(C)) \in \mathcal{H}$.*

Proof. By continuity, for every $k \in \mathbb{N}$ there is $k' \geq k$ such that $H(C \cap k') \cap k = H(C) \cap k$. So there is $(G, B) \in \mathcal{F}$ satisfying $B \cap k' = A \cap k'$ and $G(C \cap k') \cap k' = H(C \cap k') \cap k'$. So

$$H(C) \cap k = H(C \cap k') \cap k = G(C \cap k') \cap k.$$

Since k was arbitrary, we have $(C, H(C)) \in \mathcal{H}$. $\square$

Claim 7.4. *For $(H, A) \in \mathcal{F}$ and $C \subseteq A$ we have $H(C)\Delta F(C) \in \mathcal{K}$.*

Proof. Assume otherwise. Since $\mathcal{K}$ is closed, there is k such that $(H(C)\Delta F(C)) \cap k \notin \mathcal{K}$. Because H is continuous, there is $k' \geq k$ such that $(H(C \cap k')\Delta F(C)) \cap k \notin \mathcal{K}$. There is $(G, B) \in \mathcal{F}_0$ such that $D \supseteq C \cap k'$ and $F(C) \cap k = G(C \cap k') \cap k$. So $(G(C \cap k')\Delta H(C \cap k')) \cap k \notin \mathcal{K}$, and this is impossible because of the assumption on $\mathcal{F}$. $\square$

These two claims complete the proof of Lemma 7.2. $\square$

Lemma 7.5. *If $\mathcal{J}$ is a nonmeager ideal on $\mathbb{N}$ and $\mathcal{H} \subseteq \mathcal{P}(\mathbb{N})$ is such that the set*

$$\{A \in \mathcal{J} | \mathcal{H} \text{ is relatively nonmeager on } \mathcal{P}(A)\}$$

is cofinal in $\mathcal{J}$ with respect to the inclusion modulo finite, then $\mathcal{H} \cap \mathcal{J}$ is nonmeager.

Proof. If $\mathcal{H} \cap \mathcal{J}$ is meager, then there is an increasing sequence n_i $(i \in \mathbb{N})$ and $s_i \subseteq [n_i, n_{i+1})$ for $i \in \mathbb{N}$ such that the dense G_δ set

$$G = \{A \subseteq \mathbb{N} | (\exists^\infty i) A \cap [n_i, n_{i+1}) = s_i\}$$

is disjoint from $\mathcal{H} \cap \mathcal{J}$. Since $\mathcal{J}$ is nonmeager, by [21] and [37], we can find an infinite $C \subseteq \mathbb{N}$ such that $B = \bigcup_{i \in C} s_i \in \mathcal{J}$. If $A \in \mathcal{A}$ is such that $B \setminus A$ is finite, then $G \cap \mathcal{P}(A)$ is relatively comeager, and therefore $\mathcal{H}$ is not relatively nonmeager on $\mathcal{P}(A)$. □

The following replaces [11, Theorem 3.11.3].

Lemma 7.6. *Assume $\mathcal{I}$ is an ideal on $\mathbb{N}$, $\Phi \colon \mathcal{P}(\mathbb{N}) \to \mathcal{P}(\mathbb{N})/\mathcal{I}$ is a homomorphism, $\mathcal{K}$ is a closed approximation to $\mathcal{I}$, $\mathcal{J}_2(\Phi)$ is a nonmeager P-ideal, and there are Baire-measurable functions f_i $(i \in \mathbb{N})$ whose graphs cover the graph of a function g that is a $\mathcal{K}$-approximation to Φ on $\mathcal{J}_{\mathrm{cont}}(\Phi)$.*
Then there is a continuous $\mathcal{K}^3$-approximation to Φ on $\mathcal{J}_{\mathrm{cont}}(\Phi)$.

Proof. First we find functions f_i' $(i \in \mathbb{N})$ that are continuous $\mathcal{K}^2$-approximations to Φ on $\mathcal{J}_{\mathrm{cont}}(\Phi)$ as follows (this part of the proof is identical to the proof of [11, Claim 1, p. 94]). Find an increasing sequence $\{n_i\}$ of natural numbers and $s_i \subseteq [n_i, n_{i+1})$ such that each f_i is continuous on the dense G_δ set

$$G = \{A \subseteq \mathbb{N} | (\exists^\infty i) A \cap [n_i, n_{i+1}) = s_i\}.$$

Since $\mathcal{J}_{\mathrm{cont}}(\Phi)$ is nonmeager, there is an infinite $C = \{k(i) : i \in \mathbb{N}\} \subseteq \mathbb{N}$ such that $\bigcup_{i \in C} s_i \in \mathcal{J}_{\mathrm{cont}}(\Phi)$. For $\varepsilon \in \{0, 1\}$ let $A_\varepsilon = \bigcup_{i=0}^\infty [n_{k(2i+\varepsilon)}, n_{k(2i+1+\varepsilon)})$, and let $S_\varepsilon = \bigcup_{i=0}^\infty s_{k(2i+\varepsilon)}$. For $i, j \in \mathbb{N}$ define a function f_{ij}' by

$$f_{2^i(2j+1)}'(B) = f((B \cap A_0) \cup S_0) \cap \Phi_*(A_0)$$
$$\cup f((B \cap A_1) \cup S_1) \setminus \Phi_*(A_0).$$

Then each f_i' is a continuous function, and their graphs cover a graph of a $\mathcal{K}^2$-approximation to Φ on $\mathcal{J}_{\mathrm{cont}}(\Phi)$ (see [11, Claim 2, p. 94] for details).
For a function g, define

$$\mathcal{H}(g) = \{A \in \mathcal{J}_{\mathrm{cont}}(\Phi) : \Phi_*(A) \Delta g(A) \in \mathcal{K}^2 \underline{\cup} \mathcal{I}\}.$$

Then we have $\mathcal{J}_{\mathrm{cont}}(\Phi) = \bigcup_{i=0}^\infty \mathcal{H}(f_i')$. We modify the family $\{f_i'\}$ once more. If $t \in 2^{<\mathbb{N}}$, write $[t]$ for the basic open subset of $\mathcal{P}(\mathbb{N})$ determined by t. For each i and t such that $\mathcal{H}(f_i') \cap [t]$ is nonmeager, define a function $f_{i,t}''$ by ($|t|$ is the length of t, and we write $\tilde{t} = \{j | t(j) = 0\}$)

$$f_{i,t}''(A) = f_i'((A \setminus |t|) \Delta \tilde{t}) \Delta \Phi_*((A \cap |t|) \Delta \tilde{t}).$$

Then $f_{i,t}''$ (when defined) coincides with f_i' on $[t]$, the set $\mathcal{H}(f_{i,t}'')$ is everywhere nonmeager, and $\mathcal{J}_{\mathrm{cont}}(\Phi) \subseteq \bigcup_{i,t} \mathcal{H}(f_{i,t}'')$. Re-enumerate $\{f_{i,t}''\}$ as g_i $(i \in \mathbb{N})$.
For each $A \in \mathcal{J}_2(\Phi)$ let Ψ_A be a continuous almost lifting of Φ on $\mathcal{P}(A)$. Since $\mathcal{K}^2 \underline{\cup} \mathcal{I} \subseteq \mathcal{K}^3 \underline{\cup} \mathrm{Fin}$, by the Baire Category Theorem there are t_A, i_A, and k_A such that the set

$$G_A = \{B \in ([t_A] \cap \mathcal{J}_{\mathrm{cont}}(\Phi) \cap \mathcal{P}(A)) | (\Psi_A(B) \Delta g_{i_A}(B)) \setminus k_A \in \mathcal{K}^3\}$$

has a relatively nonmeager intersection with $\mathcal{J}_{\mathrm{cont}}(\Phi) \cap \mathcal{P}(A)$. Since Ψ_A and g_{i_A} are continuous and $\mathcal{K}^2$ is closed, the set G_A is closed.

Since $\mathcal{J}_2(\Phi)$ is a P-ideal, there are $\bar{t}, \bar{i}$, and $\bar{k}$ such that

$$\mathcal{A} = \{A \in \mathcal{J}_2(\Phi) | (t_A, i_A, k_A) = (\bar{t}, \bar{i}, \bar{k})\}$$

is cofinal in $\mathcal{J}_2(\Phi)$ under the inclusion modulo finite. By Lemma 7.5, the set $\mathcal{H}(g_{\bar{i}})$ is nonmeager, and it is therefore everywhere nonmeager. For each $A \in \mathcal{J}_2(\Phi)$ and $m \in \mathbb{N}$ the set

$$H_{A,m} = \{B \in \mathcal{J}_{\text{cont}}(\Phi) \cap \mathcal{P}(A) | (\Psi_A(B) \Delta g_{\bar{i}}(B)) \setminus m \notin \mathcal{K}^3\}$$

is open. Since $H_{A,m} \cap \mathcal{H}(g_{\bar{i}}) = \emptyset$, by Lemma 7.5 the set

$$X_m = \{A \in \mathcal{J}_2(\Phi) | H_{A,m} \neq \emptyset\}$$

is not cofinal in $\mathcal{J}_2(\Phi)$ under the ordering $\subseteq^*$. By [11, Lemma 2.2.2], $\bigcup_m X_m$ is cofinal in neither $(\mathcal{J}_2(\Phi), \subseteq^*)$ nor $(\mathcal{J}_2(\Phi), \subseteq)$. So its complement, the set

$$\{A \in \mathcal{J}_2(\Phi) | (\forall B \subseteq A) \Psi_A(B) \Delta g_{\bar{i}}(B) \in \mathcal{K}^3 \underline{\cup} \text{Fin}\}$$

is cofinal in $(\mathcal{J}_2(\Phi), \subseteq)$. Therefore $\mathcal{H}(g_{\bar{i}}) = \mathcal{J}_{\text{cont}}(\Phi)$. $\square$

8. Baire-measurable liftings and approximations

As in §1, for $f \colon \mathcal{P}(\mathbb{N}) \to \mathcal{P}(\mathbb{N})$ and $A, B \subseteq \mathbb{N}$ let

$$D_{A,B}^f = f(A) \Delta f(B) \Delta f(A \Delta B).$$

Let $\mathcal{C}$ denote a poset for adding a Cohen subset to $\mathbb{N}$, $\{0,1\}^{<\mathbb{N}}$, the set of all finite sequences in $\{0,1\}$ ordered by the extension. Let $\dot{c}_1$ and $\dot{c}_2$ be the canonical $\mathcal{C} \times \mathcal{C}$ names for side-by-side Cohen reals. Lemma 8.4 below was essentially proved in an unpublished paper [25] (all results of that paper are proved in [26] without appealing to this lemma). A similar method was first used in [4]. Recall that $\mathcal{K}^n$ stands for the family of sets that are unions of n elements of $\mathcal{K}$.

Note that in the following lemma we do not require x and $x \Delta z$ to be side-by-side generic.

Lemma 8.1. *Assume $\mathcal{H} \subseteq \mathcal{P}(\mathbb{N})$ is nonmeager and M is a countable transitive model of a sufficiently large fragment of ZFC. Then there is $A \in \mathcal{H}$ such that for every $z \subseteq \mathbb{N}$ the set*

$$\mathcal{X}_A(z) = \{x \subseteq A : \text{ both } x \text{ and } x \Delta z \text{ are Cohen-generic over } M\}$$

is a dense G_δ subset of $\mathcal{P}(A)$.

Proof. Let $\mathcal{D}_n$ $(n \in \mathbb{N})$ be an enumeration of all dense open subsets of $\mathcal{C}$ that belong to M. For each n fix a partition $\mathbb{N} = \bigcup_{i=1}^{\infty} I_i^n$ of $\mathbb{N}$ into finite intervals and $s_i^n \subseteq I_i^n$ such that

$$\mathcal{D}_n \supseteq \{B : (\exists^{\infty} n) B \cap I_i^n = s_i^n\}.$$

Now find a partition $\mathbb{N} = \bigcup_{i=1}^{\infty}$ of $\mathbb{N}$ into finite intervals such that J_i includes pairwise disjoint intervals $I_{k(i,j)}^j$ $(j \leq i)$ and let $t_i = \bigcup_{j=1}^{i} s_{k(i,j)}^j$. The set

$$\mathcal{Y} = \{B \subseteq \mathbb{N} | (\exists^{\infty} i) B \cap J_i = t_i\}$$

is dense G_δ, so we can find $A \in \mathcal{Y} \cap \mathcal{H}$, and A is as required. $\square$

Lemma 8.2. *Assume $\mathcal{I}$ is an analytic ideal, $\mathcal{K}$ is its analytic approximation, $\mathcal{J}$ is a nonmeager ideal including* Fin*, and f is a Borel-measurable map such that*

$D^f_{XY} \in \mathcal{I}$ for all X, Y in $\mathcal{J}$. Then there is a Borel measurable map g such that

(1) for every $A \subseteq \mathbb{N}$ there are A_1, A_2, A_3 such that
  (a) $A_1 \in \mathcal{J}$, $A \Delta A_2 \in \mathcal{J}$, and $A_3 \in \mathrm{Fin}$,
  (b) $A = A_1 \Delta A_2 \Delta A_3$, and
  (c) $g(A) = (f(A_1) \Delta f(A_2) \Delta f(A_3)) \setminus \bar{m}$;
(2) for all X, Y in $\mathcal{J}$ we have $D^g_{X,Y} \in \mathcal{K}^{22}$.

Proof. First note that the set

$$\mathcal{Y} = \{(X, Y) : D^f_{X,Y} \notin \mathcal{K} \underline{\cup} \mathrm{Fin}\}$$

is meager. Otherwise there would be a nonempty open subset of $\mathcal{P}(\mathbb{N})^2$ such that $\mathcal{Y}$ is relatively comeager on this set. So since $\mathcal{J}^2$ is closed under finite changes of its elements, it has a nonmeager intersection with every nonmeager set with the Baire property. Since $\mathcal{J}^2 \cap \mathcal{Y} = \emptyset$, $\mathcal{Y}$ is meager. Therefore $\mathcal{C} \times \mathcal{C} \Vdash D^f_{\dot{c}_1 \dot{c}_2} \in \mathcal{K} \underline{\cup} \mathrm{Fin}$.

Pick a condition $(u, v) \in \mathcal{C} \times \mathcal{C}$ that forces

$$D^f_{\dot{c}_1 \dot{c}_2} \setminus n \in \mathcal{K}$$

for some $n \in \mathbb{N}$. We may assume that u and v have the same length, and denote this length by k. Let l be such that

(3) $D^f_{s,t} \setminus l \in \mathcal{K}$ for all $s, t \subseteq k$.

Let $\bar{m} = \max(n, l)$. Replacing f with the map $A \mapsto f(A) \setminus \bar{m}$, we may assume $\bar{m} = n = l = 0$. Fix a countable transitive model M of a large enough fragment of ZFC that contains everything relevant. If $m \in \mathbb{N}$ and $x_1, x_2, \ldots, x_m$ are real numbers, then let $M[x_1, x_2, \ldots, x_m]$ denote some countable transitive model of a large enough fragment of ZFC including M and containing all x_i. The fact that this model is not unique will be irrelevant.

Claim 8.3. *If x, y, x', y' are Cohen-generic subsets of $[k, \infty)$ over M or empty and $x \Delta y = x' \Delta y'$, then*

$$D^f_{u\Delta x, u\Delta y} \Delta D^f_{u\Delta x', u\Delta y'} = f(u\Delta x) \Delta f(u\Delta y) \Delta f(u\Delta x') \Delta f(u\Delta y')$$

belongs to $\mathcal{K}^4$.

Note that we are not assuming that x, y, x' and y' are side-by-side generic.

Proof. Let $a \subseteq [k, \infty)$ be Cohen-generic over $M[x, y, x', y']$. Then

$$b = a\Delta x \Delta y = a\Delta x' \Delta y'$$

is also Cohen-generic over the same model, and all pairs $\langle x, b\rangle$, $\langle y, a\rangle$, $\langle x', a\rangle$ and $\langle y', b\rangle$, as well as their finite changes, are side-by-side Cohen-generic over M. Note that

$$D^f_{u\Delta x, u\Delta y} \Delta D^f_{u\Delta x', u\Delta y'} = D^f_{u\Delta x, v\Delta b} \Delta D^f_{u\Delta y, v\Delta a} \Delta D^f_{u\Delta x', v\Delta a} \Delta D^f_{u\Delta y', v\Delta b}.$$

Since the right-hand side belongs to $\mathcal{K}^4$, this proves the claim. $\square$

By Lemma 8.1 let $\bar{B} \subseteq \mathcal{J}$ be such that for each $z \subseteq \mathbb{N}$ let

$$\mathcal{X}(z) = \{x \subseteq \bar{B} \cap [k, \infty) : \text{ both } x \text{ and } x\Delta z \text{ are Cohen generic over } M\}$$

is a relatively comeager subset of $\mathcal{P}(\bar{B})$. Note that the set of all pairs (z, x) such that $x \in \mathcal{X}(z)$ is Borel (even G_δ). Therefore the last sentence of [28, Corollary 18.7]

implies that there is a Borel function c such that $c(A) \in \mathcal{X}(A)$ for all A. Define $g\colon \mathcal{P}(\mathbb{N}) \to \mathcal{P}(\mathbb{N})$ by

$$g(A) = f(u \cup c(A))\Delta f(u \cup (c(A)\Delta(A \setminus k)))\Delta f(A \cap k).$$

Let us check that (1) holds with $A_1 = u \cup x(A)$, $A_2 = u \cup (c(A)\Delta(A \setminus k))$ and $A_3 = A \cap k$. Clearly (1c) holds. Note that $c(A) \subseteq B$, and therefore $u \cup c(A) \in \mathcal{J}$ and $u \cup (c(A)\Delta(A \setminus k))\Delta A \in \mathcal{J}$, so (1a) holds.

Also, $A \cap k$ is finite. We have $c(A) \cap k = \emptyset$, $u \cap (A \setminus k) = \emptyset$ and

$$(u \cup c(A))\Delta(u \cup (c(A)\Delta(A \setminus k))) = A \setminus k,$$

so (1b) holds.

We claim that for $A', A'' \subseteq [k, \infty)$ we have $D^g_{A',A''}\Delta f(\emptyset) \in \mathcal{K}^{22}$. To see this, let X', X'' be side-by-side Cohen-generic reals over $M[A', A'']$ such that $X' \in \mathcal{X}(A')$ and

$$X'' \in \mathcal{X}(A'') \cap (X'\Delta\mathcal{X}(A'\Delta A'')).$$

Hence if $X = X'\Delta X''$ and $A = A'\Delta A''$, then $X \in \mathcal{X}(A)$. Let $Y' = X'\Delta A'$, $Y'' = X''\Delta A''$, and $Y = Y'\Delta Y''$. Note that $A' \setminus k = A'$ and $X'\Delta Y' = A' = c(A')\Delta(c(A')\Delta A')$, and also $f(A' \cap k) = f(\emptyset)$ (by the assumption that $l = 0$). Therefore by Claim 8.3 we have

(4) $g(A')\Delta f(u \cup X')\Delta f(u \cup Y') \in \mathcal{K}^4\Delta f(\emptyset)$.

An analogous argument shows that

(5) $g(A'')\Delta f(u \cup X'')\Delta f(u \cup Y'') \in \mathcal{K}^4\Delta f(\emptyset)$,
(6) $g(A)\Delta f(u \cup X)\Delta f(u \cup Y) \in \mathcal{K}^4\Delta f(\emptyset)$.

Since $X'\Delta X'' = X\Delta\emptyset$ and $Y'\Delta Y'' = Y\Delta\emptyset$, by Claim 8.3 we have

(7) $f(u\Delta X')\Delta f(u\Delta X'')\Delta f(u\Delta X)\Delta f(u) \in \mathcal{K}^4$,
(8) $f(u\Delta Y')\Delta f(u\Delta Y'')\Delta f(u\Delta Y)\Delta f(u) \in \mathcal{K}^4$.

Note that $D^g_{X,Y} = D^f_{X\cap k, Y\cap k}\Delta D^f_{X\setminus k, Y\setminus k}$. Since $f(\emptyset)\Delta D^g_{A',A''}$ is the symmetric difference of the left-hand sides of (4)–(8) and $f(\emptyset) \in \mathcal{K}$, this proves our claim.

By (3) and the assumption that $l = 0$ we get $D^g_{A'A''} \in \mathcal{K}^{22}$, as required. $\square$

Lemma 8.4. *Assume $\mathcal{I}$ is an analytic ideal, $\mathcal{K}$ is its analytic approximation, and f is a Borel-measurable map such that $D^f_{XY} \in \mathcal{I}$ for all X, Y. Then there is a Borel measurable map g such that, for all X and Y in $\mathcal{P}(\mathbb{N})$,*

(9) $f(X)\Delta g(X) \in \mathcal{I}$, *and*
(10) $D^g_{XY} \in \mathcal{K}^{22}$.

Proof. Take $\mathcal{J} = \mathcal{P}(\mathbb{N})$ and apply Lemma 8.2. The resulting function g is as required by (1b) and (1c). $\square$

Lemma 8.5. *If $\mathcal{I}$ is an analytic ideal and $\Phi\colon \mathcal{P}(\mathbb{N}) \to \mathcal{P}(\mathbb{N})/\mathcal{I}$ has a Baire-measurable almost lifting, then for every analytic approximation $\mathcal{K}$ to $\mathcal{I}$ there is a continuous almost lifting $f = f(\mathcal{K}, \Phi)$ of Φ such that*

$$D^f_{A,B} \in \mathcal{K}^{88}$$

for all A, B.

Proof. Let g be as guaranteed by Lemma 8.4. Let G be a dense G_δ set on which g is continuous, and let $\mathbb{N} = A_0\dot\cup A_1$ be a disjoint partition of $\mathbb{N}$ such that for some $S_0 \subseteq A_0$ and $S_1 \subseteq A_1$ we have $\{X : X \cap A_0 = S_0 \text{ or } X \cap A_1 = S_1\} \subseteq G$. As in the

proof of Lemma 6.3, we may pick S_0 and S_1 so that $S = S_0 \cup S_1$ belongs to the nonmeager ideal $\mathcal{J}$ on which g is a lifting on Φ. Let

$$f(X) = g((X \cap A_0)\Delta S)\Delta g((X \cap A_1)\Delta S).$$

Then f is a continuous lifting of Φ on $\mathcal{J}$, and

$$D^f_{X,Y} = D^g_{(X\cap A_0)\Delta S,(Y\cap A_0)\Delta S}\Delta D^g_{(X\cap A_1)\Delta S,(Y\cap A_1)\Delta S}$$
$$\Delta D^g_{(X\Delta Y)\cap A_0,(X\Delta Y)\cap A_1}\Delta D^g_{((X\Delta Y)\cap A_0)\Delta S,((X\Delta Y)\cap A_1)\Delta S}$$

belongs to $\mathcal{K}^{88}$. Therefore f is as required. $\qquad\square$

Lemma 8.6. *Assume an ideal $\mathcal{J}$ is nonmeager, $\mathcal{K}$ is closed and hereditary, and f is a $\mathcal{K}$-approximation of g on $\mathcal{J}$. Moreover, assume that there is an $m \in \mathbb{N}$ for which $D^f_{X,Y} \in \mathcal{K}^m$ and $D^g_{X,Y} \in \mathcal{K}^m$ for all X,Y. Then there is $p = p(f,g) \in \mathbb{N}$ such that*

$$(f(A)\Delta g(A)) \setminus p \in \mathcal{K}^{2m+2}$$

for all $A \subseteq \mathbb{N}$.

Proof. Assume not. Pick sequences n_i, k_i, A_i ($i \in \mathbb{N}$) such that for all i we have (let $k_0 = n_0 = 0$)

(1) $k_i \in \mathbb{N}$, $n_i \in \mathbb{N}$, $A_i \in \mathcal{P}(\mathbb{N})$,
(2) $k_i = \min\{k : (\forall s \subseteq n_{i-1})(f(s)\Delta g(s)) \setminus k \in \mathcal{K}\}$,
(3) $(f(B)\Delta g(B)) \setminus k_i \notin \mathcal{K}^{2m+2}$ for every B such that $(B\Delta A_i) \cap n_i = \emptyset$, and
(4) the sequence n_i is increasing.

Let us describe the recursive construction of k_i, n_i and A_i. Assume k_i, n_i, A_i have been found for all $i \leq l$. Since f and g are liftings of the same homomorphism, we can find k_{l+1} that satisfies (2). Since by our assumption for every k there is A such that $(f(A)\Delta g(A)) \setminus k \notin \mathcal{K}^{2m+2}$, we can find A_{l+1} such that

$$(f(A_{l+1})\Delta g(A_{l+1})) \setminus k_{l+1} \notin \mathcal{K}^{2m+2}.$$

Finally, n_{l+1} that satisfies (3) and (4) exists by the continuity of f and g and the fact that $\mathcal{K}$ is closed and hereditary.

Since $\mathcal{J}$ is nonmeager, we can find an infinite set $C \subseteq \mathbb{N}$ such that

$$A_\infty = \bigcup_{i \in C} A_i \cap [n_{i-1}, n_i)$$

belongs to $\mathcal{J}$. Since $f(A_\infty)\Delta g(A_\infty) \in \mathcal{K}\underline{\cup}\mathrm{Fin}$, there exists $i \in \mathbb{N}$ such that $(f(A_\infty)\Delta g(A_\infty)) \setminus k_i \in \mathcal{K}$. Letting $s = (A_i\Delta A_\infty) \cap n_i$ and

$$A_i^* = (A_i \cap n_i) \cup (A_\infty \setminus n_i)$$

and noting that $s \subseteq n_{i-1}$, we have the following

$$f(A_i^*)\Delta g(A_i^*) = f(A_\infty\Delta s)\Delta g(A_\infty\Delta s)$$
$$\subseteq (f(s)\Delta g(s)) \cup D^f_{s,A_\infty} \cup D^g_{s,A_\infty} \cup (f(A_\infty)\Delta g(A_\infty)).$$

By (3) the left-hand side does not belong to $\mathcal{K}^{2m+2}\underline{\cup}\{k_i\}$. But the right hand side belongs to $\mathcal{K}^{2m+2}\underline{\cup}\{k_i\}$, a contradiction. $\qquad\square$

9. The local continuous lifting property

The main result of this section is Theorem 9.10, where we prove a local version of Theorem 4 for a larger class of ideals.

Definition 9.1. Assume $\mathcal{I}$ is countably 2-determined by a sequence $\mathcal{K}_m$ ($m \in \mathbb{N}$) of its analytic approximations. We say that $\mathcal{I}$ *admits Luzin gaps with respect to $\mathcal{K}_m$ ($m \in \mathbb{N}$)* if for every gap $\mathcal{A}, \mathcal{B}$ in $\mathcal{P}(\mathbb{N})/\mathcal{I}$ and every m one of the following applies:

(a) $\mathcal{A}, \mathcal{B}$ are $\mathcal{K}_m \sqcup \mathcal{K}_m$-unapproachable, or
(b) a proper poset $\mathcal{P}$ adds a $\mathcal{K}_m$-Luzin subgap to $\mathcal{A}, \mathcal{B}$.

If there is such a sequence of analytic approximations, we say that $\mathcal{I}$ admits Luzin gaps.

Lemma 9.2. *If $\mathcal{I}$ is countably 2-determined by a sequence of closed approximations, then it admits Luzin gaps with respect to that sequence of approximations.*

Proof. This is Theorem 4.4. $\square$

Definition 9.3. An ideal $\mathcal{I}$ has the *continuous lifting property* if every homomorphism $\Phi \colon \mathcal{P}(\mathbb{N}) \to \mathcal{P}(\mathbb{N})/\mathcal{I}$ has a continuous almost lifting.

An ideal $\mathcal{I}$ has the *local continuous lifting property* if for every homomorphism $\Phi \colon \mathcal{P}(\mathbb{N}) \to \mathcal{P}(\mathbb{N})/\mathcal{I}$ the ideal

$$\mathcal{J}_{\mathrm{cont}}(\Phi) = \{A : \Phi \restriction \mathcal{P}(A) \text{ has a continuous lifting}\}$$

is linearly ccc over Fin.

Definition 9.4. If $\Phi \colon \mathcal{P}(\mathbb{N}) \to \mathcal{P}(\mathbb{N})/\mathcal{I}$ is a homomorphism, then a forcing $\mathcal{P}$ *spoils* Φ if $\mathcal{P}$ generically adds $X \subseteq \mathbb{N}$ and an uncountable family B_y, C_y ($y \in I$) of ground model sets such that

(i) $B_y \subseteq X$, $X \cap C_y = \emptyset$ for all $y \in I$, and
(ii) $\Phi_*(B_y), \Phi_*(C_y)$ ($y \in I$) form a Luzin gap over $\mathcal{I}$.

By Lemma 3.2, if Φ is spoiled, then in every further $\aleph_1$-preserving forcing extension Φ cannot be extended to a homomorphism. For a closed hereditary set $\mathcal{K}$, define a partition $[\mathcal{P}(\mathbb{N}) \times \mathcal{P}(\mathbb{N})]^2 = L_0^{\mathcal{K}} \cup L_1^{\mathcal{K}}$ by letting $\{(A, B), (C, D)\} \in L_0^{\mathcal{K}}$ if and only if

$$(A \cap D) \cup (B \cap C) \notin \mathcal{K}.$$

Note that this partition is open.

We will need a forcing notion that was introduced and used in this context already in Shelah's proof that all automorphisms of $\mathcal{P}(\mathbb{N})/\operatorname{Fin}$ are consistently trivial ([34]). If $\mathcal{F}$ is a family of pairs $s = \langle A^s, B^s \rangle$ such that for all $s \in \mathcal{F}$ we have $B^s \subseteq A^s$ and for all distinct s, t in $\mathcal{F}$ the set $A^s \cap A^t$ is finite, define the poset $\mathcal{Q}_{\mathcal{F}}$ as follows.

The conditions are all pairs $q = \langle A^q, B^q \rangle$ such that for some finite $\mathcal{F}' \subseteq \mathcal{F}$ we have (by $=^*$ and $\subseteq^*$ we denote the equality modulo finite and the inclusion modulo finite, respectively)

(1) $A^q =^* \bigcup_{s \in \mathcal{F}'} A^s$, and
(2) $B^q \cap A^s =^* B^s$, for all $s \in \mathcal{F}'$.

The ordering is defined by $p \le q$ if

(3) $A^p \supseteq A^q$ and $B^p \cap A^q = B^q$.

Note that if $\mathcal{F}$ is countable, so is $\mathcal{Q}_{\mathcal{F}}$, and that $\mathcal{Q}_{\emptyset}$ is a standard poset for adding a Cohen subset to $\mathbb{N}$. If $G \subseteq \mathcal{Q}_{\mathcal{F}}$ is a sufficiently generic filter, then $X^G = \bigcup_{p \in G} B^p$ splits the pregap $\{B^s : s \in \mathcal{F}\}$, $\{A^s \setminus B^s : s \in \mathcal{F}\}$ over Fin. Since $p \perp q$ if and only if $(B^p \Delta B^q) \cap A^p \cap A^q$ is nonempty, and this condition does not depend on $\mathcal{F}$, we will freely use this notation for pairs $p = (A^p, B^p)$ and $q = (A^q, B^q)$ such that $B^p \subseteq A^p$ and $B^q \subseteq A^q$, even if $\mathcal{Q}_{\mathcal{F}}$ is not specified. A simple consequence of this characterization of incompatibility in $\mathcal{Q}_{\mathcal{F}}$, used below, is that if p, q, and r are pairwise compatible, then a single condition extends all three of them.

Lemma 9.5. *Assume $\mathcal{F}$ is countable, $\mathcal{D}$ is a maximal antichain in $\mathcal{Q}_{\mathcal{F}}$, and A is almost disjoint modulo finite with all A^s ($s \in \mathcal{F}$). Then the set*

$$\{B \subseteq A : \mathcal{D} \text{ is a maximal antichain in } \mathcal{Q}_{\mathcal{F} \cup \{(A,B)\}}\}$$

is relatively comeager in $\mathcal{P}(A)$.

Proof. Pick a $q \in \mathcal{Q}_{\mathcal{F}}$ such that $A^q \cap A = \emptyset$. Let

$$X_q = \{B \subseteq A : (\forall p \in \mathcal{D}) \text{ if } q \not\perp p \text{ then } (A, B) \perp p\}.$$

We claim that X_q is a nowhere dense subset of $\mathcal{P}(A)$. If $U \subseteq \mathcal{P}(A)$ is relatively clopen and nonempty, then there are a finite $a \subseteq A$ and $b \subseteq a$ such that $U = \{B \subseteq A : B \cap a = b\}$. Moreover, $q_1 = (a \cup A^q, b \cup B^q)$ is a condition in $\mathcal{Q}_{\mathcal{F}}$. By the maximality of $\mathcal{D}$, there is $p \in \mathcal{D}$ compatible with q_1. Then $A^p \cap A \supseteq a$ and $B^p \cap a = b$. Since $A^p \cap A$ is finite, $\{B \subseteq A : B \cap A^p = B^p \cap A\}$ is a nonempty relatively clopen subset of $\mathcal{P}(A)$ included in U that avoids X_q.

Since $\mathcal{Q}_{\mathcal{F}}$ is countable,

$$Y = \mathcal{P}(A) \setminus \bigcap_{q \in \mathcal{Q}_{\mathcal{F}}} \{B \subseteq A : (\exists B' \in X_q) B' \Delta B \text{ is finite}\}$$

contains a comeager subset. We need to prove that $\mathcal{D}$ is a maximal antichain in $\mathcal{Q}_1 = \mathcal{Q}_{\mathcal{F} \cup \{(A,B)\}}$. Pick $q \in \mathcal{Q}_1 \setminus \mathcal{Q}_{\mathcal{F}}$, and let $q' = (A^q \setminus A, B^q \setminus A)$. Since $q \notin \mathcal{Q}_{\mathcal{F}}$, the set $(B^q \cap A) \Delta B$ is finite. Therefore $B^q \cap A \notin X_{q'}$, and there is $p \in \mathcal{D}$ compatible with q' and $(A, B^q \cap A)$. Then p is compatible with q. $\qquad\square$

Assume $\mathcal{F}$ is countable and $A \subseteq \mathbb{N}$ is almost disjoint modulo finite to all A^s ($s \in \mathcal{F}$). For $B \subseteq A$ we write

$$\mathcal{Q}_{\mathcal{F}}(B) = \mathcal{Q}_{\mathcal{F} \cup \{(A,B)\}}.$$

If $\mathcal{F}$ is clear from the context, we drop the subscript and write $\mathcal{Q}(B)$ instead of $\mathcal{Q}_{\mathcal{F}}(B)$. If $q \in \mathcal{Q}_{\mathcal{F}}$, then

$$q(B) = q \cup (\chi_B \upharpoonright (A \setminus \mathrm{dom}(q)))$$

is a condition in $\mathcal{Q}(B)$ that extends q, and

$$\mathcal{Q}'(B) = \{q(B) : q \in \mathcal{Q}_{\mathcal{F}}\}$$

is a dense subset of $\mathcal{Q}(B)$.

If $\dot{\mathcal{H}}$ is a $\mathcal{Q}_{\mathcal{F}}$ name for an analytic set, then by Lemma 9.5 there is a dense G_δ subset $G_{\dot{\mathcal{H}}} \subseteq \mathcal{P}(A)$ such that $\dot{\mathcal{H}}$ is a $\mathcal{Q}_{\mathcal{F}}(B)$-name for an analytic set for every $B \in G_{\dot{\mathcal{H}}}$. This is because $\dot{\mathcal{H}}$ is given as a projection of a tree $\dot{T}$, and a name for $\dot{T}$ is given by a countable family of maximal antichains (a 'nice name' in the sense of [30] or a 'simple name' in the sense of [2, Definition 3.6.11]).

The following lemma is well-known.

Lemma 9.6. *Using the terminology of the previous paragraph, for $r \in \mathcal{Q}_{\mathcal{F}}$ the set*

$$X = \{(B, D) : B \in G_{\dot{\mathcal{H}}}, D \in \mathbb{R}, r(B) \Vdash_{\mathcal{Q}(B)} D \in \dot{\mathcal{H}}\}$$

is analytic.

Proof. Let $\dot{T}$ be a name for a tree such that $\Vdash_{\mathcal{Q}_{\mathcal{F}}} \dot{\mathcal{H}} = p[\dot{T}]$. Then $(B, D) \in X$ if and only if there is a $\mathcal{Q}(B)$-name $\dot{b}$ for a real such that for every m and every $q(B) \leq r(B)$ there is $p(B) \leq q(B)$ such that $p \Vdash (D \restriction m, \dot{b} \restriction m) \in \dot{T}$. $\square$

The ideals $\mathcal{J}_{\sigma-}^{\mathcal{K}}(\Phi)$ were defined in §5.

Lemma 9.7. *Assume $\Phi \colon \mathcal{P}(\mathbb{N}) \to \mathcal{P}(\mathbb{N})/\mathcal{I}$ is a homomorphism and $\mathcal{K}$ is a closed approximation to $\mathcal{I}$ such that $\mathcal{J}_{\sigma-}^{\mathcal{K}^4}(\Phi)$ is not ccc over Fin. Then a proper poset spoils Φ.*

Proof. Fix a family A_α $(\alpha < \omega_1)$ of $\mathcal{J}_{\sigma-}^{\mathcal{K}^4}(\Phi)$-positive pairwise disjoint modulo finite sets. Let $\mathcal{P}_{\mathcal{D}}$ be the poset consisting of all $p = \langle \beta^p, B_\alpha^p : \alpha < \beta^p, \mathcal{D}^p \rangle$ such that

(4) $\beta^p < \omega_1$,
(5) $B_\alpha^p \subseteq A_\alpha$ for all $\alpha < \beta^p$, and
(6) $\mathcal{D}^p$ is a countable family of maximal antichains in $\mathcal{Q}_{\mathcal{F}^p}$, where

$$\mathcal{F}^p = \{(A_\alpha, B_\alpha^p) : \alpha < \beta^p\}.$$

The ordering is defined by $p \leq q$ if

(7) $\beta^p \geq \beta^q$, $B_\alpha^p = B_\alpha^q$ for all $\alpha < \beta^q$, and $\mathcal{D}^p \supseteq \mathcal{D}^q$.

Note that the last condition in particular implies that every member of $\mathcal{D}^q$ remains a maximal antichain in $\mathcal{Q}_{\mathcal{F}^p}$. If $G \subseteq \mathcal{P}_{\mathcal{D}}$ is a generic filter, for $\alpha < \omega_1$ let $B_\alpha^G = B_\alpha^p$ for some $p \in G$ such that $\beta^p > \alpha$, and let $C_\alpha^G = A_\alpha \setminus B_\alpha^p$. Note that B_α^G and C_α^G are well-defined ground-model subsets of $\mathbb{N}$. Let

$$\mathcal{F}^G = \{(A_\alpha, B_\alpha^G) : \alpha < \omega_1\}.$$

Claim 9.8. *The poset $\mathcal{P}_{\mathcal{D}}$ is σ-closed and it forces that $\mathcal{Q}_{\mathcal{F}^G}$ is ccc.*

Proof. It is clear that $\mathcal{P}_{\mathcal{D}}$ is σ-closed. Let τ be a $\mathcal{P}_{\mathcal{D}}$ name for a maximal antichain in $\mathcal{Q}_{\mathcal{F}^G}$. Since $\mathcal{P}_{\mathcal{D}}$ is σ-closed, we may find $p \in \mathcal{P}_{\mathcal{D}}$ and a maximal antichain $\mathcal{A} \subseteq \mathcal{Q}_{\mathcal{F}^p}$ such that p forces that $\mathcal{A}$ is included in $\mathrm{int}(\tau)$. Extend p by adding $\mathcal{A}$ to $\mathcal{D}^p$. Then this condition forces that $\mathrm{int}(\tau) = \mathcal{A}$, in particular that $\mathrm{int}(\tau)$ is countable. Since τ was arbitrary, this proves that $\mathcal{P}_{\mathcal{D}}$ forces $\mathcal{Q}_{\mathcal{F}^G}$ is ccc. $\square$

Claim 9.9. *The poset $\mathcal{P}_{\mathcal{D}} * \mathcal{Q}_{\mathcal{F}^G}$ adds $X \subseteq \mathbb{N}$ such that the families*

$$\mathcal{A} = \{\Phi_*(A) : A \setminus X \in \mathcal{I}, A \in V\},$$
$$\mathcal{B} = \{\Phi_*(B) : B \cap X \in \mathcal{I}, B \in V\}$$

form a gap that is not $\mathcal{K}^2$-unapproachable over $\mathcal{I}$.

Proof. Let $\dot{F}_n$, $\dot{G}_n$ $(n \in \mathbb{N})$ be $\mathcal{P}_{\mathcal{D}} * \mathcal{Q}_{\mathcal{F}^G}$-names for hereditary sets such that it is forced that $\dot{F}_n \square \dot{G}_n \subseteq \mathcal{K}^2$ for all n. We need to find $\bar{p}$ in $\mathcal{P}_{\mathcal{D}} * \mathcal{Q}_{\mathcal{F}^G}$ that forces that the interpretations of $\dot{F}_n$ and $\dot{G}_n$ $(n \in \mathbb{N})$ do not witness unapproachability of $\mathcal{A}$ and $\mathcal{B}$. More precisely, we need to find an ordinal $\gamma < \omega_1$ such that $\bar{p}$ forces that for every n we have either $\Phi_*(C_\gamma^G) \notin \dot{F}_n \underline{\cup} \mathcal{I}$ or $\Phi_*(B_\gamma^G) \notin \dot{G}_n \underline{\cup} \mathcal{I}$.

For a real x let $F_n(x)$, $G_n(x)$ ($n \in \mathbb{N}$) be the closed hereditary sets coded by x. Let $\dot{r}$ be a $\mathcal{P}_{\mathcal{D}} * \mathcal{Q}_{\mathcal{F}G}$-name for a real such that $\dot{F}_n = F_n(\dot{r})$ and $\dot{G}_n = G_n(\dot{r})$ for all n is forced. Pick a countable elementary submodel M of $H_{\mathfrak{c}^+}$ containing everything relevant, and find an $(M, \mathcal{P}_{\mathcal{D}})$-generic condition $p_0 \in \mathcal{P}_{\mathcal{D}}$ that decides a $\mathcal{Q}_{\mathcal{F}G}$-name for $\dot{r}$ (possible since $\mathcal{P}_{\mathcal{D}}$ is σ-closed and $\mathcal{Q}_{\mathcal{F}G}$ is forced to be ccc). We may assure that $p_0 \subseteq M$. Then $\mathcal{D}^{p_0}$ contains all the maximal antichains of $\mathcal{Q}_{\mathcal{F}p_0}$ that belong to M, in particular all maximal antichains occurring in $\dot{r}$.

Pick $\gamma \in \omega_1 \setminus M$. For $D \subseteq \mathbb{N}$ consider the following statement of $V^{\mathcal{Q}(B)}$:

$$\Phi_*(A_\gamma) \setminus D \in F_n(\dot{r}) \underline{\cup} \mathcal{I},$$
$$(P_n(D)) \qquad\qquad D \in G_n(\dot{r}) \underline{\cup} \mathcal{I}, \text{ and}$$
$$D \setminus \Phi_*(A_\gamma) \in \mathcal{I}.$$

Let

$$\mathcal{X} = \{B \subseteq A_\gamma : \not\Vdash_{\mathcal{Q}(B)} \neg(\exists n) P_n(\Phi_*(B))\}.$$

For $n \in \mathbb{N}$ and $q \in \mathcal{Q}_{\mathcal{F}}$ let

$$G_{n,q} = \{(B, D) : B \subseteq A_\gamma \text{ and } q(B) \Vdash_{\mathcal{Q}(B)} P_n(D)\}.$$

Then for $B \in \mathcal{X}$ there are $q \in \mathcal{Q}_{\mathcal{F}}$ and $n \in \mathbb{N}$ such that

$$q(B) \Vdash P_n(\Phi_*(B)),$$

and therefore

(8) $B \in \mathcal{X}$ if and only if $(\exists n)(\exists q)(B, \Phi_*(B)) \in G_{n,q}$.

If for some n and q we have $\{(B, D_1), (B, D_2)\} \subseteq G_{n,q}$, then

$$D_1 \Delta D_2 = (D_1 \setminus D_2) \cup (D_2 \setminus D_1) \in (F_n(\dot{r}) \underline{\cap} G_n(\dot{r}))^2 \subseteq \mathcal{K}^4 \underline{\cup} \mathcal{I}.$$

Therefore we have

(9) $(\forall B \in \mathcal{X})(\exists n)(\exists q)(\forall D)$

$$(B, D) \in G_{n,q} \to D \Delta \Phi_*(B) \in \mathcal{K}^4 \underline{\cup} \mathcal{I}.$$

By Lemma 9.6 each $G_{n,q}$ is analytic, so by [28, Theorem 18.1] it has a C-measurable uniformization $g_{n,q}$ from a subset of $\mathcal{P}(A_\gamma)$ into $\mathcal{P}(\Phi_*(A_\gamma))$. By (9), $g_{n,q}$ is a $\mathcal{K}^4$-approximation of Φ_* on $\mathcal{X}$ (see Definition 5.1). If $\mathcal{X}$ is a relatively comeager subset of $\mathcal{P}(A_\gamma)$, then the family $\{g_{n,q} : n \in \mathbb{N}, q \in \mathcal{Q}_{\mathcal{F}}\}$ witnesses that $A_\gamma \in \mathcal{J}_{\sigma-}^{\mathcal{K}^4}(\Phi)$ (since then the restriction of Φ_* to the complement of $\mathcal{X}$ is trivially Baire-measurable), contradicting the assumption. Therefore $\mathcal{P}(A_\gamma) \setminus \mathcal{X}$ is relatively nonmeager in $\mathcal{P}(A_\gamma)$. Hence by Lemma 9.5 we can find $\bar{B} \subseteq A_\gamma$ such that

(10) all antichains in $\mathcal{D}^{p_0}$ are maximal in $\mathcal{Q}(\bar{B})$, and

(11) $\Vdash_{\mathcal{Q}(\bar{B})} (\forall n) \neg P_n(\Phi_*(\bar{B}))$.

Let $p_1 \leq p_0$ be such that $B_\gamma^{p_1} = \bar{B}$ and all maximal antichains of $\mathcal{Q}(\bar{B})$ that belong to M are in $\mathcal{D}^{p_1}$.

Let $H \subseteq \mathcal{P}_{\mathcal{D}} * \mathcal{Q}_{\mathcal{F}G}$ be a generic filter containing $(p_1, 1_{\mathcal{Q}(\bar{B})})$. Then $H \cap M$ is $(M, \mathcal{P}_{\mathcal{D}} * \mathcal{Q}_{\mathcal{F}G})$-generic, because p_1' is $(M, \mathcal{P}_{\mathcal{D}})$-generic and $\mathcal{Q}_{\mathcal{F}G}$ is ccc. Note that $\bar{r} = \text{int}_H(\dot{r})$ belongs to $M[H \cap M]$. By Shoenfield's Absoluteness Theorem we have

$$(\forall n) \neg P_n(\Phi_*(\bar{B})),$$

and therefore sequences $F_{\bar{n}}(\dot{r}), G_{\bar{n}}(\dot{r})$ do not separate C_ξ ($\xi < \omega_1$) from B_ξ ($\xi < \omega_1$). Since this was an arbitrary name for a sequence of closed hereditary sets, this concludes the proof. $\square$

By Lemma 4.3 and Claim 9.9, there is a $\mathcal{P}_\mathcal{D} * \mathcal{Q}_{\mathcal{F}G}$-name $\dot{\mathcal{R}}$ for a proper poset that adds a $\mathcal{K}$-Luzin subgap to $\mathcal{A}, \mathcal{B}$ as defined in Claim 9.9. Let $\{\dot{B}_\alpha : \alpha < \omega_1\}$ be a $\mathcal{P}_\mathcal{D} * \mathcal{Q}_{\mathcal{F}G} * \dot{\mathcal{R}}$-name such that $\mathcal{A}^G, \mathcal{B}^G$ evaluated from these parameters is forced to be frozen. The poset $\mathcal{P}_\mathcal{D} * \mathcal{Q}_{\mathcal{F}G} * \dot{\mathcal{R}}$ is proper, and we need to meet only $\aleph_1$ dense sets in order to assure that the gap is $\mathcal{K}$-Luzin. Therefore PFA implies that there is $X \subseteq \mathbb{N}$ such that $\{\Phi_*(A_\alpha \setminus X) : \alpha < \omega_1\}$ and $\{\Phi_*(A_\alpha \cap X) : \alpha < \omega_1\}$ form a $\mathcal{K}$-Luzin gap in $\mathcal{P}(\mathbb{N})/\mathcal{I}$. But $\Phi_*(X)$ splits this gap—a contradiction. $\qquad\square$

Theorem 9.10 (PFA). *Assume that $\mathcal{I}$ is countably 32-generated by a sequence of analytic approximations and that it admits Luzin gaps with respect to the same sequence. Then $\mathcal{I}$ has the local continuous lifting property.*

Proof. Let $\mathcal{K}_m$ ($m \in \mathbb{N}$) be closed approximations that countably 32-determine $\mathcal{I}$ as in Definition 9.1. Assume the contrary, that $\mathcal{J}_{\mathrm{cont}}(\Phi)$ is not linearly ccc over Fin for some homomorphism $\Phi \colon \mathcal{P}(\mathbb{N}) \to \mathcal{P}(\mathbb{N})/\mathcal{I}$. By the assumption that $\mathcal{I}$ is countably 32-generated by $\mathcal{K}_m$ ($m \in \mathbb{N}$) and Lemma 5.6, there is m such that $\mathcal{J}_{\mathrm{cont}}^{(\mathcal{K}_m)^{32}}(\Phi)$ is not linearly ccc over Fin. By Lemma 5.5, Lemma 5.7, and Lemma 5.2, $\mathcal{J}_{\sigma-}^{\mathcal{K}^4}(\Phi)$ is not ccc over Fin. Therefore by Lemma 9.7 a proper poset $\mathcal{P}$ spoils Φ. Since we need to meet only $\aleph_1$ dense sets in order to witness that Φ is spoiled, Φ is not a homomorphism; a contradiction. $\qquad\square$

10. The continuous lifting property

The following is an extension of [11, Proposition 3.13.3].

Proposition 10.1. *Assume $\mathcal{I}$ is 3d-generated by closed approximations $\mathcal{K}_m$ ($m \in \mathbb{N}$), that $d \geq 178$, and that $\Phi \colon \mathcal{P}(\mathbb{N}) \to \mathcal{P}(\mathbb{N})/\mathcal{I}$ is a homomorphism. For every $D \subseteq \mathbb{N}$, each of the following conditions implies the next.*

(12) *For every m there are $\mathcal{X}_n^m$ ($n \in \mathbb{N}$) such that each $\mathcal{X}_n^m$ is $L_1^{(\mathcal{K}_m)^d}(D, n)$-homogeneous and $\mathcal{J} = \bigcup_{n=1}^\infty X_n^m$.*

(13) *$D \in \mathcal{J}_2(\Phi)$.*

(14) *For every m there are $\mathcal{X}_n^m$ ($n \in \mathbb{N}$) such that each $\mathcal{X}_n^m$ is $L_1^{(\mathcal{K}_m)^{178}}(D, 0)$-homogeneous and $\mathcal{J} = \bigcup_{n=1}^\infty X_n^m$.*

Proof. Assume (12). For simplicity of notation we assume that $D = \mathbb{N}$. For X_n^m define
$$\mathcal{H}_n^m = \{s \cup A | A \in \mathcal{X}_n^m, s \subseteq n, \text{ and } \min(A) \geq n\}.$$
Then $\bigcup_n \mathcal{H}_n^m = \mathcal{J}$ for every m. For $A \in \mathcal{H}_n^m$, modify Ψ_A as follows:
$$\Psi_A'(C) = (\Psi_A(C \setminus n) \Delta \Phi_*(C \cap n)) \setminus n.$$
Then Ψ_A' is still a continuous lifting of Φ on $\mathcal{P}(A) \cap \mathcal{J}_{\mathrm{cont}}(\Phi)$. We also claim that each $\mathcal{H}_n^m$ is $L_1^{(\mathcal{K}_m)^d}(\mathbb{N}, 0)$-homogeneous when the functions Ψ_A' are used to compute the partition. Assume A and B belong to $\mathcal{H}_n^m$, and fix $u \subseteq A \cap B$. Then
$$\Psi_A'(u) \Delta \Psi_B'(u) = (\Psi_{A \setminus n}(u \setminus n) \Delta \Psi_{B \setminus n}(u \setminus n)) \setminus n,$$
so it belongs to $(\mathcal{K}_m)^d$. By Lemma 7.2, for every m there is a $\mathcal{K}_m^d$-approximation to Φ on $\mathcal{J}_{\mathrm{cont}}(\Phi)$ whose graph is covered by graphs of countably many Baire-measurable functions. By Lemma 7.6, for every m there is a continuous $(\mathcal{K}_m)^{3d}$-approximation Θ_m of Φ on $\mathcal{J}_{\mathrm{cont}}(\Phi)$. Since $\mathcal{I} = \bigcap_m (\mathcal{K}_m)^{3d} \underline{\cup}$ Fin, by Lemma 7.1 we have $\mathbb{N} \in \mathcal{J}_2(\Phi)$.

Now assume (13) and fix m. Then Lemma 8.6 with $p = 88$ and (15) together imply that for every $B \in \mathcal{J}_2(\Phi)$ there is an $N_B \in \mathbb{N}$ such that

$$(\Psi_D(X)\Delta\Psi_B(X)) \setminus N_B \in (\mathcal{K}_m)^{178}$$

for all $X \subseteq D \cap B$. The set $\{B : N_B = \bar{N}\}$ is clearly $L_1^{(\mathcal{K}_m)^{178}}(D, 0)$-homogeneous for each $\bar{N}$, and these sets cover $\mathcal{J}_2(\Phi)$. $\qquad\square$

Let $\mathcal{K}_m$ ($m \in \mathbb{N}$) be closed approximations that 712-generate $\mathcal{I}$, and fix m for a moment. By Lemma 8.5, for each $A \in \mathcal{J}_2(\Phi)$ and each m there is a continuous lifting $\Psi_{A,m}$ of Φ on $\mathcal{J}_{\mathrm{cont}}(\Phi) \cap \mathcal{P}(A)$ such that

(15) $D_{X,Y}^{\Psi_{A,m}} \in \mathcal{K}^{88}$ for all $X, Y \subseteq A$.

By Lemma 8.6 the family $\{\Psi_{A,m} | A \in \mathcal{J}_2(\Phi)\}$ is $\mathcal{K}'_m$-coherent, with

$$\mathcal{K}'_m = (\mathcal{K}_m)^{178}.$$

In Lemma 10.2 and Theorem 10.4 below, partitions $L_0^{(\mathcal{K}_m)^d}(\mathbb{N}, k)$ are always computed with respect to functions $\Psi_{A,m}$ that are adjusted to $\mathcal{K}_m$.

Lemma 10.2 (PFA)**.** *If $\mathcal{I}$ is countably 712-generated by a sequence of its closed approximations and $\Phi \colon \mathcal{P}(\mathbb{N}) \to \mathcal{P}(\mathbb{N})/\mathcal{I}$ is a homomorphism, then $\mathcal{J}_2(\Phi)$ is a P-ideal.*

Proof. Let A_n ($n \in \mathbb{N}$) be an increasing sequence of sets in $\mathcal{J}_2(\Phi)$. We need to find $A \in \mathcal{J}_2(\Phi)$ such that $A_n \setminus A$ is finite for all n. We may assume that the domain of Φ is $\mathcal{P}(\mathbb{N}^2)$ and that $A_n = n \times \mathbb{N}$. For $f \colon \mathbb{N} \to \mathbb{N}$ let

$$\Gamma_f = \{(m, n) : n \leq f(m)\}.$$

There is $\bar{f}$ such that for all g we have $\Gamma_g \setminus \Gamma_{\bar{f}} \in \mathcal{J}_{\mathrm{cont}}(\Phi)$. Otherwise we could find a $\leq^*$-increasing sequence $\{f_\xi : \xi < \omega_1\}$ such that $\Gamma_{f_{\xi+1}} \setminus \Gamma_{f_\xi}$ is $\mathcal{J}_{\mathrm{cont}}(\Phi)$-positive for all ξ, contradicting the fact that $\mathcal{J}_{\mathrm{cont}}(\Phi)$ is linearly ccc over Fin (Theorem 9.10). Since we only need to find g such that $\mathbb{N}^2 \setminus \Gamma_g \in \mathcal{J}_2(\Phi)$, for simplicity we may assume that $\bar{f}(n) = 0$ for all n; consequently $\Gamma_g \in \mathcal{J}_{\mathrm{cont}}(\Phi)$ for all g.

We claim that $\mathcal{X} = \{\Gamma_f : f \in \mathbb{N}^{\mathbb{N}}\}$ has no uncountable $L_0^{\mathcal{K}'_m}(\mathbb{N}^2, 0)$-homogeneous subsets. Assume the contrary, and let $\mathcal{H}$ be such. Since OCA implies that every subset of $\mathbb{N}^{\mathbb{N}}$ of size $\aleph_1$ is bounded ([38, Theorem 3.4]), we can find $A \in \mathcal{X}$ such that the set $\mathcal{X} \cap \mathcal{P}(A)$ is uncountable. This implies that $\mathcal{X} \cap \mathcal{P}(A)$ is also $L_0^{\mathcal{K}'_m}(A, 0)$-homogeneous, contradicting Proposition 10.1. Therefore OCA implies that $\mathcal{X}$ can be covered by $L_1^{\mathcal{K}'_m}(\mathbb{N}, 0)$-homogeneous sets $\mathcal{X}_n$ ($n \in \mathbb{N}$). One of these sets, call it $\mathcal{F}_m$, is cofinal in $\mathbb{N}^{\mathbb{N}}/\mathrm{Fin}$ (see, e.g., [11, Lemma 2.2.2]).

Claim 10.3. *For every $B \in \mathcal{J}_2(\Phi)$ there is a $k = k_m(B) \in \mathbb{N}$ such that for all $f \in \mathcal{F}_m$*

$$\{B, \Gamma_f\} \in L_1^{(\mathcal{K}_m)^{356}}(\mathbb{N}^2 \setminus A_k, k).$$

We choose each $k_m(B)$ large enough so that for every $g \in \mathbb{N}^{\mathbb{N}}$ there is $f \in \mathcal{F}_m$ such that $g(i) \leq f(i)$ for all $i \geq k$.

Proof. By Lemma 8.6 with $p = 88$ and (15), for every $f \in \mathcal{F}_m$ there is $p(f) \in \mathbb{N}$ such that for all $X \subseteq B \cap \Gamma_f$ we have

$$(\Psi_{B,m}(X)\Delta\Psi_{f,m}(X)) \setminus p(f) \in (\mathcal{K}_m)^{178}.$$

Let $\bar{k} \in \mathbb{N}$ be such that $\mathcal{F}'_m = \{f \in \mathcal{F}_m : p(f) \leq \bar{k}\}$ is cofinal in $\mathbb{N}^{\mathbb{N}}$ modulo finite. Let $\bar{l}$ be such that for every $g \in \mathbb{N}^{\mathbb{N}}$ there is $f \in \mathcal{F}'_m$ such that $f(i) \geq g(i)$ for all $i \geq \bar{l}$. We claim that $k = \max(\bar{k}, \bar{l})$ is as required.

Pick $g \in \mathcal{F}_m$, and find $f \in \mathcal{F}'_m$ as above. Fix $X \subseteq (B \cap \Gamma_g) \setminus A_k \subseteq (B \cap \Gamma_f)$. We have

$$\Psi_{B,m}(X)\Delta\Psi_{g,m}(X) = (\Psi_{B,m}(X)\Delta\Psi_{f,m}(X))\Delta(\Psi_{f,m}(X)\Delta\Psi_{g,m}(X)).$$

By the definition of $\mathcal{F}'_m$ and its $L_1^{(\mathcal{K}_m)^{178}}(\mathbb{N}^2, 0)$-homogeneity, the right hand side belongs to $(\mathcal{K}_m)^{356} \sqcup \{k\}$, and this completes the proof. $\quad\square$

For $k \in \mathbb{N}$ let

$$\mathcal{H}_{m,k} = \{C : (\exists A \in \mathcal{J}_2(\Phi))C \subseteq A \text{ and } k_m(A) = k\}.$$

For $C \in \mathcal{H}_{m,k}$ and $A \supseteq C$ such that $k_m(A) = k$, we may replace $\Psi_{C,m}$ with the restriction of $\Psi_{A,m}$ to $\mathcal{P}(C)$. Hence we have $k_m(C) = k$ for all $C \in \mathcal{H}_{m,k}$. Moreover, if $C \in \mathcal{H}_{m,k}$, we can replace $\Psi_{C,m}$ by the map

$$D \mapsto (\Psi_{A_k}(D \cap A_k) \cap \Psi_{A_k}(A_k)) \cup (\Psi_{B,m}(D \setminus A_k) \setminus \Psi_{A_k}(A_k)).$$

If A, B are in $\mathcal{H}_{m,k}$ and $s \subseteq A \cap B$, then by Claim 10.3 (b) there is $f \in \mathcal{F}_m$ such that $\Gamma_f \supseteq s \setminus A_k$. Thus, after the partition is re-evaluated using the new functions, we have

(16) Each $\mathcal{H}_{m,k}$ is $L_1^{(\mathcal{K}_m)^{712}}(\mathbb{N}, 0)$-homogeneous.

By Proposition 10.1 we have $\mathbb{N}^2 \in \mathcal{J}_2(\Phi)$, concluding the proof of Lemma 10.2. $\quad\square$

Theorem 10.4 (PFA). *Every $\mathcal{I}$ that is countably 3204-determined by closed approximations has the continuous lifting property.*

Proof. Let $\mathcal{K}_m$ ($m \in \mathbb{N}$) be closed approximations that 3204-generate $\mathcal{I}$, and fix a homomorphism $\Phi \colon \mathcal{P}(\mathbb{N}) \to \mathcal{P}(\mathbb{N})/\mathcal{I}$. By Theorem 9.10, $\mathcal{J}_2(\Phi)$ is ccc over Fin and by Lemma 10.2 it is a P-ideal.

Fix m. Assume for a moment that there is no sequence $\mathcal{X}_n^m$ ($n \in \mathbb{N}$) such that each $\mathcal{X}_n^m$ is $L_1^{(\mathcal{K}_m)^{1068}}(\mathbb{N}, n)$-homogeneous, and that $\mathcal{J}_{\text{cont}}(\Phi) = \bigcup_{n=1}^{\infty} \mathcal{X}_n^m$. If $\mathcal{K}'_m = (\mathcal{K}_m)^{178}$, then by Theorem 1.8 there is an uncountable family $\mathcal{A}$ of sets that are pairwise almost disjoint modulo finite, and for every $D \in \mathcal{A}$ there is an uncountable $L_0^{\mathcal{K}'_m}(D, 0)$-homogeneous subset of $\mathcal{J}_2(\Phi)$. By Proposition 10.1 we have $D \notin \mathcal{J}_2(\Phi)$, contradicting the fact that $\mathcal{J}_2(\Phi)$ is ccc over Fin.

So for every m the ideal $\mathcal{J}_2(\Phi)$ can be covered by $L_1^{(\mathcal{K}_m)^{1068}}(\mathbb{N}, n)$-homogeneous sets $\mathcal{X}_n^m$ ($n \in \mathbb{N}$) such that $\mathcal{J}_{\text{cont}}(\Phi) = \bigcup_{n=1}^{\infty} \mathcal{X}_n^m$. Since the $\mathcal{K}_m$ ($m \in \mathbb{N}$) 3204-generate $\mathcal{I}$, by Proposition 10.1 we have $\mathbb{N} \in \mathcal{J}_2(\Phi)$, and Φ has a continuous almost lifting. $\quad\square$

We can now prove Theorem 4.

Corollary 10.5 (PFA). *If $\mathcal{I}$ and $\mathcal{J}$ are analytic ideals and $\mathcal{I}$ is strongly countably determined by closed approximations, then every isomorphism between their quotients has a continuous lifting.*

Proof. Fix an isomorphism $\Phi \colon \mathcal{P}(\mathbb{N})/\mathcal{J} \to \mathcal{P}(\mathbb{N})/\mathcal{I}$. By Theorem 10.4, Φ has a continuous almost lifting. By Lemma 6.2, Φ has a continuous lifting. $\quad\square$

By Theorem 10.4 and Lemma 2.4 we have Corollary 10.6 below. Clause (1) was proved in [11, OCA lifting theorem] using the weaker assumption OCA and MA. Results (2) and (3) are new, although Just ([23]) has proved that OCA implies that F_σ ideals have the local continuous lifting property. He has also proved ([22]) the consistency of the statement 'all analytic ideals have the local continuous lifting property.'

Corollary 10.6 (PFA).   (1) *All analytic P-ideals have the continuous lifting property.*
   (2) *All F_σ ideals have the continuous lifting property.*
   (3) *The ideals* $\mathrm{NWD}(\mathbb{Q})$, $\mathrm{NULL}(\mathbb{Q})$ *and* $\mathcal{Z}_W$ *have the continuous lifting property.*   □

11. COMPLETE BOOLEAN ALGEBRAS EMBEDDABLE INTO ANALYTIC QUOTIENTS

If $\mathcal{I}$ is an analytic ideal, then $\mathcal{P}(\mathbb{N})/\mathcal{I}$ is not a complete Boolean algebra. This is because this quotient includes a family of pairwise incompatible elements of size continuum; hence not every subset of this family has a least upper bound (see [24]). CH implies that every Boolean algebra of size at most $2^{\aleph_0}$ embeds into every analytic quotient, because under this assumption $\mathcal{P}(\mathbb{N})/\mathrm{Fin}$ is saturated, in the model-theoretic sense. Not much is known about complete Boolean algebras that are embeddable into some analytic quotient without any additional set-theoretic assumptions. Every σ-centered algebra easily embeds into every analytic quotient. It was conjectured by A. Dow that PFA implies that every complete Boolean algebra embeddable into $\mathcal{P}(\mathbb{N})/\mathrm{Fin}$ is σ-centered. It should be noted that there are ccc subalgebras of $\mathcal{P}(\mathbb{N})/\mathrm{Fin}$ that are not σ-centered, by a result of M. Bell [3]. Under OCA the Lebesgue measure algebra does not embed into $\mathcal{P}(\mathbb{N})/\mathrm{Fin}$ ([7]), but it always embeds into the quotient over the ideal $\mathcal{Z}_0$ of asymptotic density zero sets ([17, p. 491]). The results of this chapter are well-known in the case when $\mathcal{I} = \mathrm{Fin}$; see [38] for a historical background. By $<_{\mathrm{Lex}}$ we denote the lexicographical ordering on 2^{ω_1}.

Theorem 11.1 (PFA). *If $\mathcal{I}$ admits Luzin gaps, then*
   (a) *$\mathcal{P}(\omega_1)$ does not embed into $\mathcal{P}(\mathbb{N})/\mathcal{I}$, and*
   (b) *$(2^{\omega_1}, <_{\mathrm{Lex}})$ does not embed into $\mathbb{N}^{\mathbb{N}}/\mathcal{I}$.*

Proof. We will prove only (a), since the proof of (b) is almost identical. Assume $\Phi\colon \mathcal{P}(\omega_1) \to \mathcal{P}(\mathbb{N})/\mathcal{I}$ is an embedding. In $V^{\mathrm{Coll}(\omega_1,\mathfrak{c})}$ (i.e., the forcing extension by σ-closed poset that collapses $\mathfrak{c}$ to ω_1) we have $\mathfrak{c} < 2^{\omega_1}$; hence by counting arguments there is $X \subseteq \omega_1$ such that the family $A_\alpha = \Phi(X \cap \alpha)$, $B_\alpha = \Phi(\alpha \setminus X)$ $(\alpha < \omega_1)$ is a gap that is not unapproachable. Let $\mathcal{P}$ be a proper poset that adds a Luzin subgap to A_α, B_α $(\alpha < \omega_1)$. Applying PFA to $\mathrm{Coll}(\omega_1,\mathfrak{c}) * \mathcal{P}$, we can find an $\bar{X} \subseteq \omega_1$ such that the family A_α, B_α $(\alpha < \omega_1)$ as defined above includes a Luzin gap. But this gap is separated by $\Phi(\bar{X})$—a contradiction.   □

Corollary 11.2 (PFA). *If $\mathcal{I}$ is an F_σ ideal or an analytic P-ideal or $\mathrm{NWD}(\mathbb{Q})$ or $\mathrm{NULL}(\mathbb{Q})$ or $\mathcal{Z}_W$, then every complete Boolean algebra $\mathcal{B}$ that embeds into $\mathcal{P}(\mathbb{N})/\mathcal{I}$ is ccc.*

Proof. By Theorem 4.4, it suffices to prove the statement assuming $\mathcal{I}$ is countably 2-determined by a sequence of closed approximations. If $\mathcal{B}$ is complete and has an

uncountable antichain, then $\mathcal{P}(\omega_1)$ embeds into $\mathcal{B}$. Hence the conclusion follows from Theorem 11.1.   □

For an ultrafilter $\mathcal{U}$ let $f \leq_{\mathcal{U}} h$ if $\{n : f(n) \leq h(n)\} \in \mathcal{U}$. For $g \in \mathbb{N}^{\mathbb{N}}$ let $\mathcal{N}_{g,\mathcal{U}}$ be $\{f/\mathcal{U} : f \leq_{\mathcal{U}} g\}$ ordered by $\leq_{\mathcal{U}}$. The statement that there is no strictly increasing map from $\mathcal{N}_{g,\mathcal{U}}$ into $\mathbb{N}^{\mathbb{N}}/\mathrm{Fin}$ implies that every homomorphism from $C([0,1])$ into a commutative Banach algebra is automatically continuous ([6]).

Corollary 11.3 (PFA). *If $\mathcal{I}$ is an analytic P-ideal or an F_σ ideal or $\mathrm{NWD}(\mathbb{Q})$ or $\mathrm{NULL}(\mathbb{Q})$ or $\mathcal{Z}_W$, then for every nonprincipal ultrafilter $\mathcal{U}$ on $\mathbb{N}$ and every unbounded monotonic $g \in \mathbb{N}^{\mathbb{N}}$ the linear ordering $\mathcal{N}_{g,\mathcal{U}}$ does not embed into $\mathbb{N}^{\mathbb{N}}/\mathcal{I}$.*

Proof. By a result of Woodin, Martin's Axiom implies that there is a strictly increasing map from $(2^{\omega_1}, <_{\mathrm{Lex}})$ into $\mathcal{N}_{g,\mathcal{U}}$ for every nonprincipal ultrafilter $\mathcal{U}$ on $\mathbb{N}$ (see [6] or [38]). Thus if there is a strictly increasing map from $\mathcal{N}_{g,\mathcal{U}}$ into $\mathbb{N}^{\mathbb{N}}/\mathcal{I}$, then there is a strictly increasing map from $(2^{\omega_1}, <_{\mathrm{Lex}})$ into $\mathbb{N}^{\mathbb{N}}/\mathcal{I}$, contradicting Theorem 11.1.   □

12. THE RADON–NIKODÝM PROPERTY
OF THE IDEALS $\mathrm{NWD}(\mathbb{Q})$ AND $\mathrm{NULL}(\mathbb{Q})$

Recall that an ideal $\mathcal{I}$ has the Radon–Nikodým property (RNP) if every homomorphism $\Phi \colon \mathcal{P}(\mathbb{N})/\mathrm{Fin} \to \mathcal{P}(\mathbb{N})/\mathcal{I}$ has a completely additive lifting. Recall that a lifting is *completely additive* if it is of the form

$$A \mapsto \bigcup_{n \in A} s_n$$

for some sequence $\{s_n\}$ of pairwise disjoint subsets of $\mathbb{N}$. It was conjectured by Stevo Todorčević ([41]) that every homomorphism between quotients over analytic P-ideals that has a continuous lifting has a completely additive lifting. This conjecture predated Conjecture 1 and motivated the author's work on this topic.

Many, but not all, analytic ideals have the Radon–Nikodým property ([11, Chapter 1], [26], [27]).

Theorem 12.1. *Both ideals $\mathrm{NWD}(\mathbb{Q})$ and $\mathrm{NULL}(\mathbb{Q})$ have the Radon–Nikodým property.*

Proof. Let $\mathcal{I} \in \{\mathrm{NULL}(\mathbb{Q}), \mathrm{NWD}(\mathbb{Q})\}$ and let $\Phi \colon \mathcal{P}(\mathbb{N}) \to \mathcal{P}(\mathbb{N})/\mathcal{I}$ be a homomorphism with a continuous lifting f. For $U \subseteq \mathbb{Q}$ we denote by Φ^U the homomorphism of $\mathcal{P}(\mathbb{N})$ into $\mathcal{P}(U)/\mathcal{I}$ with lifting $A \mapsto f(A) \cap U$. Let μ be the usual Haar measure on $\mathcal{P}(\mathbb{Q})$.

Claim 12.2. *For every nonempty basic clopen $U \subseteq \mathbb{Q}$ there is a nonempty basic clopen $V \subseteq U$ such that Φ^V has a Borel measurable lifting g such that $D^g_{XY} = \emptyset$ for all X, Y.*
Moreover, if $\mathcal{I} = \mathrm{NULL}(\mathbb{Q})$, then we can ensure that $\mu(V) = \mu(U)/2$.

Proof. Pick a condition $(u,v) \in \mathcal{C} \times \mathcal{C}$ that decides a nonempty $V \subseteq U$ such that $D^f_{\dot{c}_1 \dot{c}_2} \cap V = \emptyset$. (If $\mathcal{I} = \mathrm{NULL}(\mathbb{Q})$, also make sure that $\mu(V) = \mu(U)/2$.) By Lemma 8.4 applied with $\mathcal{K} = \mathcal{P}(\mathbb{Q} \setminus V)$, we can find g as required.   □

Using Claim 12.2, recursively find pairwise disjoint clopen sets V_n such that $\bigcup_n V_n$ is open dense (and of full measure, if $\mathcal{I} = \mathrm{NULL}(\mathbb{Q})$) and for each n Φ^{V_n} has a Borel measurable lifting g_n that is a group homomorphism. Since $A \in \mathcal{I}$ if

and only if $A \cap V_n \in \mathcal{I}$ for all n, the map $g(X) = \bigcup_n g_n(X)$ is a Borel measurable lifting of Φ that is a group homorphism.

By [12, Proposition 4.1], there are finite sets $s_p \subseteq \mathbb{N}$ for $p \in \mathbb{Q}$ such that

$$g(X) = \{p : |X \cap s_p| \text{ is odd}\}.$$

Claim 12.3. *For every nonempty basic clopen $U \subseteq \mathbb{Q}$ there is a nonempty basic clopen $V \subseteq U$ such that Φ^V has a Borel measurable lifting h that is a Boolean algebra homomorphism.*

Moreover, if $\mathcal{I} = \mathrm{NULL}(\mathbb{Q})$, then we ensure that $\mu(V) = \mu(U)/2$.

Proof. Define

$$U_{XY}^g = g(X \setminus Y) \cap g(Y \setminus X).$$

It will suffice to find a continuous h so that $U_{XY}^h = D_{XY}^h = \emptyset$ for all X, Y. Since $U_{XY}^g \in \mathcal{I}$ for all X, Y, we can find $(u, v) \in \mathcal{C} \times \mathcal{C}$ and a large $V \subseteq U$ such that $(u, v) \Vdash U_{\dot{c}_1 \dot{c}_2}^g \cap V = \emptyset$. We may assume that $\mathrm{supp}(u) = \mathrm{supp}(v) = n$ for some n. If there is a $p \in V$ such that $s_p \setminus n$ has more than one element, then we can extend (u, v) to decide that $p \in U_{XY}^g$. Thus $|s_p \setminus n| \leq 1$ for all $p \in V$, and hence the map $X \mapsto g(X \setminus n) \cap V$ is a Boolean algebra homomorphism. Since n is finite, we can easily find a Borel measurable lifting h of Φ^V that is a Boolean homomorphism. $\square$

The required lifting is found using Claim 12.3 in the same manner we have used Claim 12.2. $\square$

For a set $X \subseteq \mathbb{N} \times \mathcal{P}(\mathbb{N})$ let $X_n = \{A : (n, A) \in X\}$ and $X^A = \{n : (n, A) \in X\}$. An ideal $\mathcal{I}$ is an *F-ideal* ([26]) if for every $\varepsilon > 0$ and $X \subseteq \mathbb{N} \times \mathcal{P}(\mathbb{N})$ such that X_n is Haar-measurable of measure at least ε the set

$$\{A : X^A \notin \mathcal{I}\}$$

is Haar-measurable and has measure at least ε.

In [26] it was proved that all F-ideals have the Radon–Nikodým property. If $\mathcal{I}$ is an analytic P-ideal, then it is an F-ideal if and only if it is nonpathological, and it is unknown whether there is a pathological analytic P-ideal that has the Radon–Nikodým property (this is [11, Question 1.14.2]).

Proposition 12.4. *There is an $F_{\sigma\delta}$ ideal that has the Radon–Nikodým property but is not an F-ideal.*

Proof. We only need to check that $\mathrm{NWD}(\mathbb{Q})$ is not an F-ideal. We may think of $\mathbb{Q}$ as $2^{<\mathbb{N}}$, the set of all finite sequences of $\{0, 1\}$, with the topology induced from $\mathcal{P}(\mathbb{N})$. Let $Y \subseteq \mathcal{P}(\mathbb{N})$ be a nowhere dense set of measure $1/2$, and let $X = \bigcup_{s \in \mathbb{Q}} \{s\} \times \{s \Delta A : A \in Y\}$. Then every vertical section of X has measure $1/2$, but if $A \subseteq \mathbb{N}$, then $X^A = \{A \Delta B : B \in Y\}$ is nowhere dense. $\square$

The following lemma is well-known, but we reproduce it for the convenience of the reader.

Lemma 12.5. *If an isomorphism Φ between quotients over two ideals on $\mathbb{N}$ has a completely additive lifting and its inverse has a completely additive lifting, then Φ has a lifting of the form $A \mapsto h^{-1}(A)$ for some h that is a Rudin–Keisler isomorphism between the underlying ideals.*

Proof. Fix $\Phi\colon \mathcal{P}(\mathbb{N})/\mathcal{I} \to \mathcal{P}(\mathbb{N})/\mathcal{J}$. Let g_1 and g_2 be the functions such that the maps $A \mapsto g_1^{-1}(A)$ and $A \mapsto g_2^{-1}(A)$ are liftings of Φ and Φ^{-1}, respectively. Let $B = \{n : g_1(g_2(n)) \neq n\}$. By a standard argument (see, e.g., [5, Lemma 9.1]) we can find sets B_1, B_2, B_3 such that $B_i \cap g_1^{-1}g_2-1B_i = \emptyset$ for $i \leq 3$. But $g_1^{-1}g_2^{-1}(C)\Delta C \in \mathcal{I}$ for all C, and therefore $B_i \in \mathcal{I}$ for $i \leq 3$. Therefore the restriction of g_2 to $\mathbb{N} \setminus B$ is a Rudin–Keisler isomorphism between $\mathcal{I}$ and $\mathcal{J}$. $\square$

13. A SEPARATION PRINCIPLE

The present author asked whether quotients over simple analytic P-ideals satisfy some natural separation principle ([11, Question 5.13.9]) or have analytic Hausdorff gaps ([11, Question 5.13.7]). It turns out that the two possibilities do not exclude each other. In Theorem 13.2 we will prove a separation principle that applies to every analytic P-ideal, and in [14] it was proved that the quotient over every F_σ P-ideal other than Fin contains an analytic Hausdorff gap.

Definition 13.1. Consider Fin as a tree under the ordering $\sqsubseteq$ of end-extension. If $\Sigma \subseteq$ Fin and $t \in \Sigma$ then

$$S_\Sigma(t) = \{s : t \sqsubseteq s \text{ and there is no } u \in \Sigma \text{ such that } t \sqsubseteq u \sqsubseteq s\},$$

is the set of *immediate successors* of t in Σ. Note that $s \in S_\Sigma(t)$ can be arbitrarily long. A node $t \in \Sigma$ is *infinitely branching* if $S_\Sigma(t)$ is infinite. A set $\Sigma \subseteq$ Fin is a *superperfect subtree* if every $t \in \Sigma$ is infinitely branching. A set $\Sigma \subseteq$ Fin is a *superperfect $\mathcal{B}$-tree* if it is a superperfect subtree of $\langle \text{Fin}, \sqsubseteq \rangle$ such that for every $s \in \Sigma$ the set

$$\bigcup\{t : \max(s) < \min(t), s \cup t \in S_\Sigma(s)\}$$

is included in some set in $\mathcal{B}$.

Theorem 13.2. *Assume $\mathcal{I}$ is countably determined by its closed approximations, $\mathcal{K}_m$ ($m \in \mathbb{N}$). Assume $\mathcal{A}$ and $\mathcal{B}$ are hereditary $\mathcal{I}$-orthogonal families and that $\mathcal{A}$ is analytic. Then one of the following applies:*

(a) $\mathcal{A}$ and $\mathcal{B}$ are unapproachable over $\mathcal{I}$.
(b) There is a superperfect $\mathcal{B}$-tree with all branches in $\mathcal{A} \setminus \mathcal{I}$.

The following is an immediate consequence of Theorem 13.2 and Lemma 3.5.

Theorem 13.3 (Todorčević, [39]). *If $\mathcal{A}$ and $\mathcal{B}$ are Fin-orthogonal and $\mathcal{A}$ is analytic, then one of the following applies:*

(a) $\mathcal{A}$ and $\mathcal{B}$ are countably separated over Fin,
(b) There is a superperfect $\mathcal{B}$-tree with all branches in $\mathcal{A}$. $\square$

Proof. Let us recall some notions from [11, §5.5]. In a game $G(\mathcal{A}, \mathcal{B})$ for $\mathcal{A}, \mathcal{B} \subseteq \mathcal{P}(\mathbb{N})$ two players, I and II, play according to the following rules:

I	$s_1 \in$ Fin, $B_1 \in \mathcal{B}$		$s_2 \in$ Fin, $B_2 \in \mathcal{B}$		$\ldots$
II		$k_1 \in \mathbb{N}$		$k_2 \in \mathbb{N}$	

It is additionally required that $s_{n+1} \subseteq \mathcal{B}_n \cap [k_n, \infty)$ for all $n \in \mathbb{N}$. If both players obey the rules, player I wins if and only if $\bigcup_n s_n$ is in $\mathcal{A}$.

Lemma 13.4. *Assume $\mathcal{A}$ and $\mathcal{B}$ are families of sets of integers. There is a superperfect $\mathcal{B}$-tree all of whose branches lie in $\mathcal{A}$ if and only if I has a winning strategy in $G(\mathcal{A}, \mathcal{B})$.*

Proof. See [11, Lemma 5.5.4]. □

Let T be a tree on Fin × Fin (i.e., the Cartesian product of Fin with itself, **not** the Fubini ideal Fin × Fin) such that $\mathcal{A} = p[T]$. For $(s, t) \in$ Fin × Fin let

$$T(s, t) = \{(u, v) \in T : (s, t) \sqsubseteq (u, v)\},$$
$$\mathcal{A}(s, t) = p[T(s, t)].$$

Fix $m \in \mathbb{N}$, and let

$$T_0^m = \{(s, t) \in T : \mathcal{A}(s, t) \text{ and } \mathcal{B} \text{ are } \mathcal{K}_m\text{-unapproachable over } \mathcal{I}\}.$$

Then $T \setminus T_0^m$ is downwards closed, and $p[T \setminus T_0^m]$ is empty if and only if $(\langle \rangle, \langle \rangle) \in T_0^m$ if and only if $\mathcal{A}$ and $\mathcal{B}$ are m-unapproachable.

Claim 13.5. $T \setminus T_0^m$ *has no terminal nodes.*

Proof. Assume $(s, t) \in T$ is such that the set S of its immediate successors is included in T_0. Thus $\mathcal{A}(u, v)$ and $\mathcal{B}$ are $\mathcal{K}_m$-unapproachable over $\mathcal{I}$ for every $(u, v) \in S$. Since S is countable, $\mathcal{A} = \bigcup_{(u,v) \in S} \mathcal{A}(u, v)$ is $\mathcal{K}_m$-unapproachable to $\mathcal{B}$ over $\mathcal{I}$. Thus (s, t) is not a (terminal) node of $T \setminus T_0^m$. □

If (a) holds, we are done. Otherwise, $\mathcal{A}$ and $\mathcal{B}$ are not $\mathcal{K}_{\bar{m}}$-unapproachable for some $\bar{m} \in \mathbb{N}$. By Lemma 13.4 we need only describe a winning strategy for I in $G(\mathcal{A} \setminus \mathcal{I}, \mathcal{B})$. By the above, $(\langle \rangle, \langle \rangle) \notin T_0^{\bar{m}}$. Let $T_1 = T \setminus T_0^{\bar{m}}$.

If $\mathcal{X}$ is a hereditary set and $B \subseteq \mathbb{N}$, let us write $B \perp_m \mathcal{X}$ for

$$(\forall k)(\exists A \in \mathcal{X})(A \cap B) \setminus k \notin \mathcal{K}_m.$$

A winning strategy for I is as follows: In the first inning, I picks $(s_1, t_1) \in T_1$ and $B_1 \in \mathcal{B}$ so that

$$B_1 \perp_{\bar{m}} \mathcal{A}(s_1, t_1).$$

This is possible (for an arbitrary $(s_1, t_1) \in T_1$), since otherwise for each $B \in \mathcal{B}$ there would be a $k_B \in \mathbb{N}$ such that $\mathcal{A}(s_1, t_1) \sqcap \{B \setminus k_B\} \in \mathcal{K}_{\bar{m}}$; thus the sets $G_{\bar{m}} = \overline{\{B \setminus k_B : B \in \mathcal{B}\}}$ and $F_{\bar{m}} = \overline{\mathcal{A}(s_1, t_1)}$ would witness that $\mathcal{A}(s_1, t_1)$ and $\mathcal{B}$ are $\mathcal{K}_{\bar{m}}$-aloof, contrary to our assumption.

Then I plays s_1, B_1. After II plays some k_1, I finds $A_1 \in \mathcal{A}(s_1, t_1)$ such that $(A_1 \cap B_1) \setminus k_1 \notin \mathcal{K}_{\bar{m}}$, and picks a finite $s_2 \subseteq (A_1 \cap B_1) \setminus \max(k_1, \max(s_1))$ such that $s_2 \notin \mathcal{K}_{\bar{m}}$. There is a t_2 such that $(s_2, t_2) \in T_1(s_1, t_1)$; hence by the above argument there is $B_2 \in \mathcal{B}$ such that

$$B_2 \perp_{\bar{m}} \mathcal{A}(s_2, t_2).$$

Then I plays s_2, B_2. After II responds with some k_2, I finds $(s_3, t_3) \in T_1(s_2, t_2)$ and B_3 such that $\min(s_3) > \max(k_2, \max(s_2))$ and

$$B_3 \perp_{\bar{m}} \mathcal{A}(s_3, t_3),$$

and plays s_3, B_3, and so on.

We have described a strategy for I such that he plays s_n, B_n ($n \in \mathbb{N}$) obeying the rules of the game, while he also picks an auxiliary sequence t_n ($n \in \mathbb{N}$) such that (s_n, t_n) ($n \in \mathbb{N}$) is a branch of T. Thus $A = \bigcup_{n=1}^{\infty} s_n$ is in $\mathcal{A} = p[T]$. Since the s_n are pairwise disjoint and $s_n \notin \mathcal{K}_{\bar{m}}$ for all n, we also have $A \notin \mathcal{I}$.

Thus the described strategy is winning for I, and by Lemma 13.4 there is a superperfect $\mathcal{B}$-tree with all of its branches in $\mathcal{A} \setminus \mathcal{I}$. □

Corollary 13.6. *Assume $\mathcal{I}$ is strongly countably determined by closed approximations, $\mathcal{A} \subseteq \mathcal{P}(\mathbb{N})$ is analytic, and $\mathcal{B} \subseteq \mathcal{P}(\mathbb{N})$ is orthogonal to $\mathcal{A}$ modulo $\mathcal{I}$ and countably directed under the inclusion modulo $\mathcal{I}$. Then $\mathcal{A}, \mathcal{B}$ does not include a Luzin gap over $\mathcal{I}$.*

Proof. Assume that $\mathcal{A}, \mathcal{B}$ contains a Luzin gap. Therefore they are not unapproachable over $\mathcal{I}$. By Theorem 13.2, there is a superperfect $\mathcal{B}$-tree all of whose branches are positive and in $\mathcal{A}$. Since $\mathcal{B}$ is σ-directed under the inclusion modulo $\mathcal{I}$, there is $B \in \mathcal{B}$ such that for every $s \in \Sigma$ the set

$$\bigcup \{t : \max(s) < \min(t), s \cup t \in S_\Sigma(s)\} \setminus B$$

belongs to $\mathcal{I}$. Therefore we can find an infinite branch A of Σ that is included in B. But this is impossible because $A \in \mathcal{A} \setminus \mathcal{I}$. $\square$

14. Preservation and freezing

By [41], Theorem 13.3 implies that all Hausdorff gaps in $\mathcal{P}(\mathbb{N})/\operatorname{Fin}$ are preserved by those embeddings of $\mathcal{P}(\mathbb{N})/\operatorname{Fin}$ into analytic quotients that have a Baire-measurable lifting. It also implies that all gaps are preserved by such embeddings of $\mathcal{P}(\mathbb{N})/\operatorname{Fin}$ into quotients over analytic P-ideals. By [14], Hausdorff gaps are not preserved by embeddings of quotients over F_σ P-ideals into other quotients over F_σ P-ideals with Baire-measurable liftings.

Theorem 14.1. *Let $\mathcal{I}$ be an analytic P-ideal, let $\mathcal{A}, \mathcal{B}$ be a gap in $\mathcal{P}(\mathbb{N})/\mathcal{I}$, and let $\Phi \colon \mathcal{P}(\mathbb{N})/\mathcal{I} \to \mathcal{P}(\mathbb{N})/\mathcal{J}$ be an embedding with a Baire-measurable lifting.*

  (1) *Luzin gaps are preserved by Baire embeddings into quotients over analytic P-ideals.*
  (2) *Luzin gaps are not preserved by Baire embeddings into quotients over F_σ ideals.*
  (3) *If $\mathcal{A}, \mathcal{B}$ is a Luzin gap and at least one of its sides is σ-directed modulo $\mathcal{I}$, then $\Phi'' \mathcal{A}, \Phi'' \mathcal{B}$ is a gap in $\mathcal{P}(\mathbb{N})/\mathcal{J}$.*
  (4) Fin *is the only F_σ P-ideal (up to the Rudin–Keisler isomorphism) such that Baire embeddings of its quotient into quotients over other analytic P-ideals preserve all Hausdorff gaps.*

Before starting the proof, we need to recall some facts about liftings.

Definition 14.2. Let $\{n_i\}_{i=1}^\infty$ be a strictly increasing sequence in $\mathbb{N}$. A sequence $\{s_j\}_{j=1}^\infty$ of finite subsets of $\mathbb{N}$ is *separated* by $\{n_i\}_{i=1}^\infty$ if for every j there is an i such that

$$\max(s_j) < n_i \leq \min(s_{j+1}).$$

A map $f \colon \mathcal{P}(\mathbb{N}) \to \mathcal{P}(\mathbb{N})$ is *asymptotically additive* if there is a strictly increasing sequence $\{n_i\}_{i=1}^\infty$ in $\mathbb{N}$ such that whenever $\{s_j\}_{j=1}^\infty$ is separated by $\{n_i\}_{i=1}^\infty$ we have

$$F\left(\bigcup_{j=1}^\infty s_j\right) = \bigcup_{j=1}^\infty F(s_j).$$

Asymptotically additive liftings were introduced in [11, Definition 1.5.1]; this definition not identical to the present one, but it is straightforward to check that the two are equivalent.

Theorem 14.3. *If $\mathcal{J}$ is an analytic P-ideal and $\Phi\colon \mathcal{P}(\mathbb{N}) \to \mathcal{P}(\mathbb{N})/\mathcal{J}$ is a homomorphism which has a Baire-measurable lifting, then it has an asymptotically additive lifting.*

Proof. The case when the kernel of Φ includes Fin was stated and proved in [11, Theorem 1.5.2], but this additional assumption was never used in the proof. $\square$

It should be noted that if $\mathcal{I}$ is an analytic ideal that is not a P-ideal, then there is a homomorphism of $\mathcal{P}(\mathbb{N})/$ Fin into $\mathcal{P}(\mathbb{N})/\mathcal{I}$ that has a completely additive lifting but not an asymptotically additive lifting. Therefore the notion of an asymptotically additive lifting is mostly relevant to homomorphisms between quotients over P-ideals. The following lemma is implicit in [11].

Lemma 14.4. *Assume F is an asymptotically additive lifting of an embedding of the quotient over $\mathcal{I} = \mathrm{Exh}(\phi_{\mathcal{I}})$ into the quotient over $\mathcal{J} = \mathrm{Exh}(\phi_{\mathcal{J}})$. Then for every $\varepsilon > 0$ we have*

$$\lim_{k\to\infty} \inf\{\phi_{\mathcal{J}}(\Psi_H(s)) : \min s \geq k, \phi_{\mathcal{I}}(s) \geq \varepsilon\} > 0.$$

Proof. Assume otherwise, and let $\{n_i\}_{i=1}^{\infty}$ be the witnessing sequence for F. Then we can recursively pick s_j $(j \in \mathbb{N})$ such that for every j

  (i) $\phi_{\mathcal{I}}(s_j) \geq \varepsilon$,
  (ii) $\{s_j\}_{j=1}^{\infty}$ is separated by $\{n_i\}_{i=1}^{\infty}$,
  (iii) $\phi_{\mathcal{J}}(s_j) < 1/j^2$.

By (i) we have $\bigcup_n s_n \notin \mathrm{Exh}(\phi_{\mathcal{I}}) = \mathcal{I}$, while by (ii) and (iii) we have $F(\bigcup_j s_j) = \bigcup_j F(s_j) \in \mathcal{J}$, contradicting the assumption that Φ is an embedding. $\square$

Proof of Theorem 14.1. (1) First recall that, by [41], if $\mathcal{P}(\mathbb{N})/\mathcal{I}$ is Baire-embeddable into a quotient over an analytic P-ideal, then $\mathcal{I}$ is an analytic P-ideal.

Fix analytic P-ideals $\mathcal{I} = \mathrm{Exh}(\phi_{\mathcal{I}})$, $\mathcal{J} = \mathrm{Exh}(\phi_{\mathcal{J}})$ and an m-Luzin gap A_x, B_x $(x \in I)$ in $\mathcal{P}(\mathbb{N})/\mathcal{I}$. Let Φ be an embedding of the Boolean algebra $\mathcal{P}(\mathbb{N})/\mathcal{I}$ into the Boolean algebra $\mathcal{P}(\mathbb{N})/\mathcal{J}$. By Theorem 14.3, Φ has an asymptotically additive lifting f. By Lemma 14.4, there are $l > 0$ and $k \in \mathbb{N}$ such that for $s \subseteq [k,\infty)$ satisfying $\phi_{\mathcal{I}}(s) > 1/m$ we have $\phi_{\mathcal{J}}(f(s)) > 1/l$. For $u, v \subseteq k$ let

$$I_{u,v} = \{x \in I : A_x \cap k = u \text{ and } B_x \cap k = v\},$$
$$\mathcal{A}_{u,v} = \{A_x : x \in I_{u,v}\},$$
$$\mathcal{B}_{u,v} = \{B_x : x \in I_{u,v}\}.$$

We will prove that the image of each pair $\mathcal{A}_{u,v}, \mathcal{B}_{u,v}$ under f is an l-Luzin gap. Fix u, v and distinct $x, y \in I_{u,v}$. Then $\phi_{\mathcal{I}}(A_x \cap B_y) > 1/m$ implies

$$\phi_{\mathcal{J}}(f(A_x) \cap f(B_y)) > 1/l.$$

Since $\mathcal{A}_{u,v}, \mathcal{B}_{u,v}$ is an m-Luzin gap in $\mathcal{P}(\mathbb{N})/\mathcal{I}$, it follows that $\{f(A_x) : x \in I_{u,v}\}$, $\{f(B_x) : x \in I_{u,v}\}$ is an l-Luzin gap in $\mathcal{P}(\mathbb{N})/\mathcal{J}$.

(2) A Luzin gap in $\mathcal{P}(\mathbb{N})/$ Fin such that both of its sides $\mathcal{A}, \mathcal{B}$ are closed subsets was constructed in [39, p. 57]. Let $\mathcal{J}_1$ be the F_σ ideal generated by Fin and $\mathcal{A}$, and let $\mathcal{J}_2$ be the F_σ ideal generated by Fin and $\mathcal{B}$, and let $\mathcal{J} = \mathcal{J}_1 \oplus \mathcal{J}_2$. (Here $\mathcal{J}_1 \oplus \mathcal{J}_2$ is the ideal on $\mathbb{N} \times \{0,1\}$ defined by $A \in \mathcal{J}_1 \oplus \mathcal{J}_2$ if and only if $\{m \in \mathbb{N}|(m,0) \in A\} \in \mathcal{J}_1$ and $\{m \in \mathbb{N}|(m,1) \in A\} \in \mathcal{J}_2$.) Then $\mathcal{P}(\mathbb{N})/$ Fin can be easily Baire-embedded into $\mathcal{P}(\mathbb{N})/\mathcal{J}$ so that the image of $\mathcal{A}, \mathcal{B}$ is separated (see [41, Example 1]).

(3) Fix an analytic P-ideal $\mathcal{I} = \mathrm{Exh}(\phi)$, an analytic ideal $\mathcal{J}$, and a Luzin gap $\mathcal{A} = \{A_x : x \in I\}$, $\mathcal{B} = \{B_x : x \in I\}$ in $\mathcal{P}(\mathbb{N})/\mathcal{I}$ such that $\mathcal{I}$ is σ-directed. Let Φ be an embedding of $\mathcal{P}(\mathbb{N})/\mathcal{I}$ into $\mathcal{P}(\mathbb{N})/\mathcal{J}$ that has a Baire-measurable lifting, Φ_*. By Lemma 6.3, we may assume Φ_* is continuous. We will prove a stronger form of (2), that no analytic hereditary set separates $\Phi''\mathcal{A}$ and $\Phi''\mathcal{B}$ over $\mathcal{J}$. Assume the contrary, that $\mathcal{C}$ is such a set, and let

$$\mathcal{A}' = \{A \subseteq \mathbb{N} : \Phi_*(A) \in \mathcal{C}\underline{\cup}\mathcal{J}\}.$$

Then $\mathcal{A}'$ is analytic, being a continuous preimage of an analytic set. Since $\mathcal{A} \subseteq \mathcal{A}'$, $\mathcal{A}'$ and $\mathcal{B}$ are not unapproachable over $\mathcal{I}$. By Theorem 13.2, there is a superperfect $\mathcal{B}$-tree T with all of its branches in $\mathcal{A}'\setminus\mathcal{I}$. Let B_n $(n \in \mathbb{N})$ be all the branchings of T. Since $\mathcal{B}$ is σ-directed modulo $\mathcal{I}$, we may pick $B \in \mathcal{B}$ such that $D_n = B_n \setminus B \in \mathcal{I}$ for all n. Now let $D \in \mathcal{I}$ be such that $D_n \setminus D$ is finite for all n. If we replace T by $T' = \{s \setminus D : s \in T\}$, then T' is still a superperfect $\mathcal{B}$-tree with all of its infinite branches in $\mathcal{A} \setminus \mathcal{I}$, and its branchings are $B'_n = B_n \setminus D$. We can pick an infinite branch of T' disjoint from all D_n that is included in B; but this contradicts the assumption that $\mathcal{A}$ and $\mathcal{B}$ are $\mathcal{I}$-orthogonal, and completes the proof of (3).

The positive part of (4) was proved in [41]. In [14] we have proved that Fin is the only F_σ P-ideal that does not have analytic Hausdorff gaps in its quotient. A construction using such gap similar to one in the proof of (2) can be used to conclude the proof of (4).   □

15. Problems

The motivation for this paper comes from the following open problem ([15]).

Question 15.1. *Does PFA imply that all analytic ideals have the (local) continuous lifting property?*

The simplest ideal for which Question 15.1 is open is $\mathrm{Fin} \times \mathrm{Fin}$, the ideal of all subsets of $\mathbb{N}^2$ that have at most finitely many infinite vertical sections. Solving Question 15.1 for this ideal would make considerable progress towards a solution to Question 15.1 for an arbitrary analytic ideal. Note that Conjecture 3 is true for $\mathrm{Fin} \times \mathrm{Fin}$, by [26]. The consistency of the local continuous lifting property for all analytic ideals can be proved by using Shelah's oracle chain condition. The consistency of the continuous lifting property is open, even for $\mathrm{Fin} \times \mathrm{Fin}$.

By Theorem 4, a positive answer to the following question would solve the $F_{\sigma\delta}$ case of Question 15.1.

Question 15.2. *Is every $F_{\sigma\delta}$ ideal strongly countably determined by closed sets?*

It is even unknown whether every $F_{\sigma\delta}$ ideal $\mathcal{I}$ has an F_σ approximation $\mathcal{K}$ such that $\mathcal{K}\underline{\cup}\mathcal{K}$ is meager (see [33]). A negative answer would imply that $\mathcal{I}$ is not countably determined by closed approximations. This is because an intersection of countably many nonmeager hereditary sets closed under finite changes is nonmeager (see, e.g., [11, Theorem 3.10.1]), and it therefore cannot have the Baire Property.

A positive answer to the following question for those ideals that admit Luzin gaps was given in §11.

Question 15.3. *Does PFA imply that every complete Boolean algebra embeddable into some analytic quotient has to be ccc?*

Problem 15.4. Characterize complete Boolean algebras that can be embeded into $\mathcal{P}(\mathbb{N})/\mathcal{I}$, for some analytic P-ideal $\mathcal{I}$. In particular, do this for $\mathcal{I} = \mathcal{I}_{1/n}$ and $\mathcal{I} = \mathcal{Z}_0$.

By Theorem 11.1, a positive answer to Question 15.5 would imply a positive answer to Question 15.3. It would also represent progress towards solving Question 15.1 (see §§9 and 10).

Question 15.5. *Does every analytic ideal admit Luzin gaps?*

In this paper we have emphasized applications of gaps to Problem 15.1, but gaps in analytic quotients per se are well-studied objects likely to have other applications ([33], [41], [40]).

Question 15.6. *Is every aloof Hausdorff gap in a quotient over an analytic P-ideal included in an analytic Hausdorff gap? Is this true assuming OCA or PFA?*

Question 15.7. *Do OCA and MA (or PFA) imply that there are no (ω_2, ω_1)-gaps in any quotient over an analytic P-ideal?*

By [41] there is an (ω_1, ω_1)-gap in every analytic quotient. By [14] OCA and MA imply that every quotient over an F_σ P-ideal except $\mathcal{P}(\mathbb{N})/\operatorname{Fin}$ has (ω_2, ω_2)-gaps. Therefore an answer to Question 15.7 would complete the picture of gap spectra of F_σ P-ideals.

A positive answer to Question 15.6 would imply the positive answer to Question 15.7 by Theorem 4.2 and the proof of [11, Corollary 5.10.3]. An answer to Question 15.7 would, together with the main result of [14], give a complete description of linear gap spectra of F_σ P-ideals under OCA and MA (or PFA). The problem of determining the (linear) gap spectra of all analytic quotients was posed by Todorčević in [41, Problem 2]. One possible approach to Question 15.7 would be via the following.

Question 15.8. *Assume $\mathcal{I}$ is an F_σ (Borel, analytic) ideal, and that the chain $A_\alpha \in \mathcal{I}$ ($\alpha < \omega_1$) is increasing modulo finite. Is there a ccc poset that adds $A \in \mathcal{I}$ such that $A_\alpha \setminus A \in \operatorname{Fin}$ for all α?*

A positive answer to Question 15.8 would imply a positive answer to Question 15.7, since then every (ω_2, ω_1)-gap would have to be aloof under OCA. It should be noted that there are a compact hereditary set $\mathcal{K}$ and an unbounded ω_1-chain inside $\mathcal{K}$ that cannot be extended ([19]). Question 15.8 has a positive answer in the case of P-ideals, and a negative answer for the coanalytic ideal of all scattered subsets of the rationals.

Acknowledgments

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